NANYANG TECHNOLOGICAL UNIVERSITY
SEMESTER I EXAMINATION 2015–2016
MH1810 – Mathematics 1

NOVEMBER 2015 TIME ALLOWED: 2 HOURS

Matriculation Number:

Seat Number:

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains SEVEN (7) questions and comprises SEVENTEEN (17) pages, including an Appendix.
2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
3. This **IS NOT** an **OPEN BOOK** exam. However, a list of formulae is provided in the attachments.
4. Candidates may use calculators. However, they should write down systematically the steps in the workings.
5. All your solutions should be written in this booklet within the space provided after each question. If you use an additional answer book, attach it to this booklet and hand them in at the end of the examination.

For examiners only

Questions	Marks
1 (15)	
2 (10)	
3 (10)	
4 (10)	

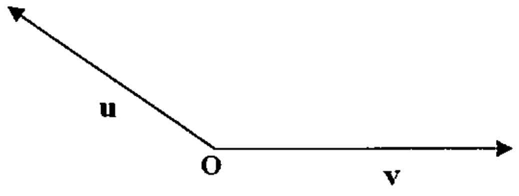
Questions	Marks
5 (15)	
6 (15)	
7 (25)	

Total (100)	

QUESTION 1. **(15 Marks)**

(a) The diagram below shows two vectors **u** and **v**. Draw **on the same diagram** the vectors

(a) (i) $\frac{1}{2}\mathbf{u}+\frac{1}{2}\mathbf{v}$
(ii) $\mathbf{u}-(\mathbf{u}\cdot\hat{\mathbf{v}})\,\hat{\mathbf{v}}$



(b) Given that the shortest distance from the point $A\,(1,2,\alpha)$ to the x-axis is 5 units. Find the possible values of α.

Question 1 continues on Page 3.

(c) Find the determinant of the following matrix by cofactor expansion via the first row. Express your answer in terms of a.

$$M = \begin{pmatrix} 1 & 0 & 0 & 1 \\ a & 0 & 0 & 0 \\ 0 & a & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

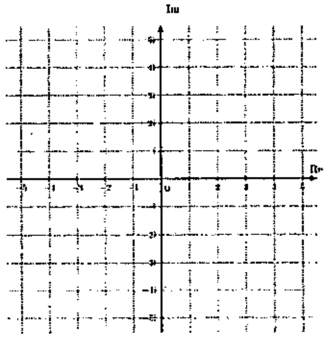
Hence, determine the value(s) of a if the matrix is singular.

QUESTION 2. **(10 Marks)**

(a) Given that $w = \dfrac{3+9i}{1-2i}$.

(i) Find $|w|$ and $\arg(w)$. Express w as $re^{i\theta}$, $r > 0$, $-\pi < \theta \leq \pi$.

(ii) Plot the points w and $-iw$ in the diagram below.



Question 2 continues on Page 5.

(b) Solve the equation

$$z^4 + z^3 + z^2 + z + 1 = 0.$$

Express the answers in the form $re^{i\theta}$, $r > 0, -\pi < \theta \leq \pi$.

QUESTION 3. **(10 Marks)**

Evaluate the following limits.

(a) $\lim\limits_{x \to -\infty} \dfrac{x^3}{\sqrt{x^6 + x + e^{-x}}}$

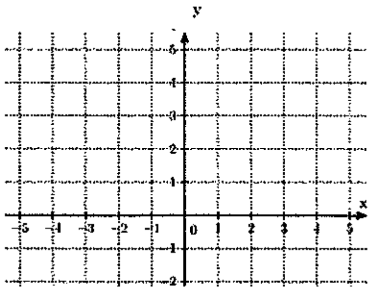
(b) $\lim\limits_{x \to \infty} \dfrac{\sin(e^x)}{1 + x^2}$

QUESTION 4. **(10 Marks)**

Consider the function f defined as follows:

$$f(x) = \begin{cases} 2x-1 & \text{if } 1 \le x \le 3 \\ |x| & \text{if } -1 < x < 1 \\ 1 & \text{if } -5 \le x \le -1 \end{cases}$$

(a) Sketch the graph of $y = f(x)$ for $-5 \le x \le 3$ in the diagram below.



(b) Evaluate the integral

$$\int_{-2}^{2} f(x)\ dx.$$

QUESTION 5 **(15 Marks)**

(a) Use the definition of derivative to find $f'(x)$, where $f(x) = x|x|$.

Question 5 continues on Page 9.

(b) Find the global maximum and minimum value of $f(x) = x^{1/3}(8-x)$ on the interval $[0,8]$.

QUESTION 6. **(15 Marks)**

(a) Assume that a snowball melts so that its volume decreases at a rate proportional to its surface area. If it takes 8 hours to melt down to $\frac{1}{8}$ of its orignal volume, how much time does it take for the snowball to melt completely?

[Volume of a sphere: $V = \frac{4}{3}\pi r^3$; Surface area of a sphere: $A = 4\pi r^2$.]

Question 6 continues on Page 11.

(b) Apply Newton's method to the equation $x^2 - a = 0$ to derive the following square-root algorithm to approximate $\sqrt{a}$:

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right).$$

Hence, starting with $x_0 = 1$, find the approximation x_2 to $\sqrt{2}$.

QUESTION 7. **(25 Marks)**

(a) Evaluate the following integrals.

(i) $\int \tan^{-1}\frac{1}{x}dx$

(ii) $\int \frac{x}{x^2+6x+10}dx$

Question 7 continues on Page 13.

(b) (i) If a and b are positive numbers, show that

$$\int_0^1 x^a (1-x)^b \, dx = \int_0^1 x^b (1-x)^a \, dx.$$

(ii) Prove that

$$\int_0^1 x (1-x)^{1/3} \, dx = \frac{9}{28}.$$

Question 7 continues on Page 14.

(c) A wedge is cut out of a circular cylinder of radius 3 by two planes. One plane is perpendicular to the axis of the cylinder. The other plane intersects the first plane at an angle 60° along a daimeter of the cylinder. Find the volume of the wedge.

END OF PAPER

Appendix

Numerical Methods.

- Linearization Formula:

$$L(x) = f(a) + f'(a)\,(x-a)$$

- Newton's Method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

- Trapezoidal Rule:

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\left[y_0 + 2\,(y_1 + y_2 + \cdots + y_{n-1}) + y_n\right]$$

- Simpson's Rule:

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\left[y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n\right]$$

Derivatives.

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(e^x) = e^x \qquad \frac{d}{dx}(a^x) = a^x \ln a$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x} \qquad \frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\cosh x) = \sinh x$$

$$\frac{d}{dx}(\tanh x) = \mathrm{sech}^2 x \qquad \frac{d}{dx}(\coth x) = -\mathrm{csch}^2 x$$

$$\frac{d}{dx}(\mathrm{sech}\ x) = -\mathrm{sech} x \tanh x \qquad \frac{d}{dx}(\mathrm{csch}\ x) = -\mathrm{csch} x \coth x$$

$$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{x^2+1}}$$

Antiderivatives.

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C, \, n \neq -1 \qquad \int \frac{1}{x} \, dx = \ln|x| + C$$

$$\int \sin x \, dx = -\cos x + C \qquad \int \cos x \, dx = \sin x + C$$

$$\int \sec^2 x \, dx = \tan x + C \qquad \int \csc^2 x \, dx = -\cot x + C$$

$$\int \tan x \sec x \, dx = \sec x + C \qquad \int \cot x \csc x \, dx = -\csc x + C$$

$$\int \tan x \, dx = \ln|\sec x| + C \qquad \int \cot x \, dx = \ln|\sin x| + C$$

$$\int e^x \, dx = e^x + C \qquad \int a^x \, dx = \frac{a^x}{\ln a} + C, \, a > 0$$

$$\int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1} x + C \qquad \int \frac{1}{1+x^2} \, dx = \tan^{-1} x + C$$

$$\int \frac{1}{\sqrt{a^2-x^2}} \, dx = \sin^{-1}\left(\frac{x}{a}\right) + C, \, |x| < |a| \qquad \int \frac{1}{x^2+a^2} \, dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{\sqrt{x^2+1}} \, dx = \sinh^{-1} x + C \qquad \int \frac{1}{\sqrt{x^2+a^2}} \, dx = \sinh^{-1}\left(\frac{x}{a}\right) + C$$

MH1810 MATHEMATICS 1

Please read the following instructions carefully:

1. **Please do not turn over the question paper until you are told to do so. Disciplinary action may be taken against you if you do so.**

2. You are not allowed to leave the examination hall unless accompanied by an invigilator. You may raise your hand if you need to communicate with the invigilator.

3. Please write your Matriculation Number on the front of the answer book.

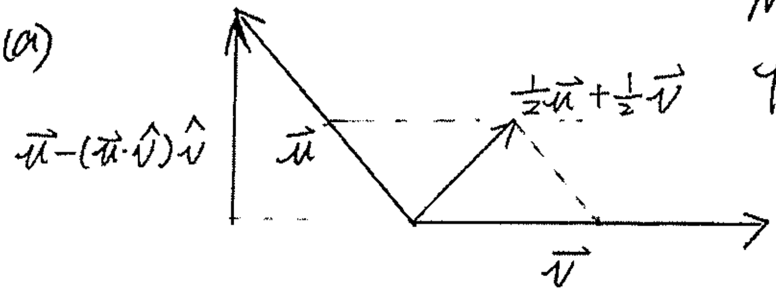
4. Please indicate clearly in the answer book (at the appropriate place) if you are continuing the answer to a question elsewhere in the book.

Subject Code	:	MH1810
Subject Name	:	Mathematatics 1
Year / Semester	:	Y15-16 / Sem 1

Suggested Solution Brought to You by Materials Science and Engineering Club

Question 1

(a)



Note: $(\vec{u}\cdot\hat{v})\hat{v}$ is the projection vector of $\vec{u}$ on $\vec{v}$

(b) $\overrightarrow{OA}=(1,2,\alpha)$ Assume $\hat{n}=(1,0\,0)$ is a unit vector on x axis

$$\text{distance}=|\overrightarrow{OA}\times\hat{n}|$$

$$=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\1&2&\alpha\\1&0&0\end{vmatrix}=|\alpha\hat{j}-2\hat{k}|=\sqrt{\alpha^2+4}$$

$$\therefore\sqrt{\alpha^2+4}=25\Rightarrow\underline{\alpha=\pm\sqrt{21}}\ \#$$

c) $\det(M)=a_{11}C_{11}+a_{14}C_{14}$

checkerboard

$$\begin{vmatrix}+&-&+&-\\-&+&-&+\\+&-&+&-\\-&+&-&+\end{vmatrix}\quad\begin{vmatrix}+&-&+\\-&+&-\\+&-&+\end{vmatrix}$$

$$=1\times\begin{vmatrix}0&0&0\\a&1&0\\1&1&1\end{vmatrix}-1\times\begin{vmatrix}a&0&0\\0&a&1\\0&1&1\end{vmatrix}$$

$$=-a\begin{vmatrix}a&1\\1&1\end{vmatrix}=-a^2+a$$

If the matrix is singular, $\det(M)=0$

$$-a^2+a=0$$

$$a=0\text{ or }a=1$$

Question 2

(a) $w=3\cdot\dfrac{(1+3i)(1+2i)}{(1-2i)(1+2i)}=3\cdot\dfrac{5i-5}{5}=3i-3=-3+3i$

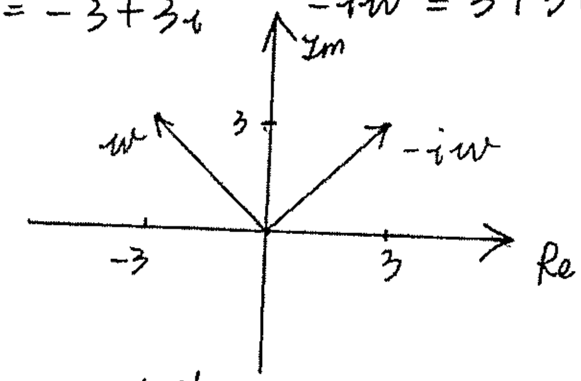
(i) $|w|=\sqrt{9+9}=2\sqrt{3}$ #

$\tan\left|\frac{y}{x}\right| = 1$ Since w is in the second quandrand

$\arg(w) = \underline{135^\circ}$ #

$w = \underline{3\sqrt{2}\, e^{\frac{3}{4}\pi}}$ #

(ii) $w = -3+3i$ $\quad -iw = 3+3i$



(b) We know that

$z^5 - 1 = (z-1)(z^4+z^3+z^2+z+1)$

The solution of equation $z^4+z^3+z^2+z+1$ is included in the solution of $z^5-1=0$

So we just need to solve $z^5=1$ and except $z=1$

$1 = e^{0\cdot i}$

The 5th root of 1 is

$z = e^{\frac{2k\pi}{5} i}$, $k = 0, 1, 2, 3, 4$

$z_1 = 1$

$z_2 = e^{\frac{2}{5}\pi i}$

$z_3 = e^{\frac{4}{5}\pi i}$

$z_4 = e^{\frac{6}{5}\pi i}$

$z_5 = e^{\frac{8}{5}\pi i}$

$\Rightarrow$ z_2, z_3, z_4 and z_5 are the solutions of the original equation.

Subject Code :	**Suggested Solution Brought to You by Materials Science and Engineering Club**
Subject Name :	
Year / Semester :	

Question 3

(a) $$\lim_{x\to-\infty}\frac{x^3}{\sqrt{x^6+x+e^{-x}}} = \lim_{x\to-\infty}\frac{x^3\cdot\frac{1}{x^3}}{\sqrt{x^6+x+e^{-x}}\cdot\frac{1}{x^3}} = \lim_{x\to-\infty}\frac{1}{\sqrt{1+x^{-5}+\frac{e^{-x}}{x^6}}}$$

$$= \lim_{x\to-\infty}\frac{1}{\sqrt{1+\frac{e^{-x}}{x^6}}} = \frac{1}{\sqrt{1+\lim_{x\to-\infty}\frac{e^{-x}}{x^6}}}$$

$$\lim_{x\to-\infty}\frac{e^{-x}}{x^6} = \lim_{x\to-\infty}\frac{-e^{-x}}{6x^5} = \lim_{x\to-\infty}\frac{e^{-x}}{30x^4} = \lim_{x\to-\infty}\frac{e^{-x}}{720} = \infty$$

(L'Hospital Rule)

$$\therefore \frac{1}{\sqrt{1+\lim_{x\to\infty}\frac{e^{-x}}{x^6}}} = 0$$

(b) $$\lim_{x\to\infty}\frac{\sin(e^x)}{1+x^2}$$

$$\because -1 \le \sin(e^x) \le 1$$

$$\therefore -\frac{1}{x^2+1} \le \frac{\sin(e^x)}{1+x^2} \le \frac{1}{1+x^2}$$

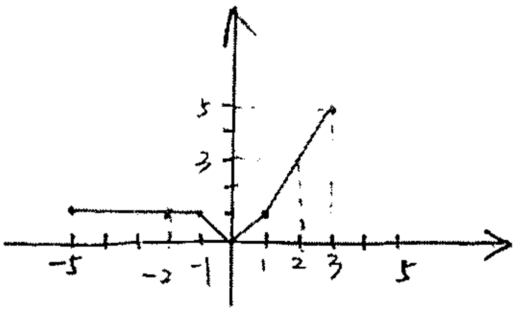
$$\because \lim_{x\to\infty}\left(-\frac{1}{1+x^2}\right) = \lim_{x\to\infty}\left(\frac{1}{1+x^2}\right) = 0$$

$\therefore$ By squeeze theorem,

$$\lim_{x\to\infty}\frac{\sin(e^x)}{1+x^2} = 0$$

Question 4

(a)



$x \in [-2, 2]$

(b) $\int_{-2}^{2} f(x)dx$ = area under $f(x)$.

Area $= 1+\frac{1}{2}+\frac{1}{2}+2 = 4$

$$\therefore \int_{-2}^{2} f(x)dx = 4$$

Question 5

(a) ① If $x>0$ then $f(x)=x^2$

$$f'(x) = \lim_{h\to 0}\frac{(x+h)^2-x^2}{h} = \lim_{h\to 0}\frac{2xh+h^2}{h} = \lim_{h\to 0}(2x+h) = 2x$$

② If $x<0$ then $f(x)=-x^2$

$$f'(x) = \lim_{h\to 0}\frac{-(x+h)^2+x^2}{h} = \lim_{h\to 0}\frac{-2xh-h^2}{h} = -2x$$

③ If $x=0$

$$\lim_{x\to 0^+} f(x) = x^2 = 0 ,\ \lim_{x\to 0^-} f(x) = -x^2 = 0 ,\ f(0)=0$$

∴ $f(x)$ is continous at $x=0$

$$f'_-(x) = \lim_{x\to 0^-}\frac{x^2-0^2}{x} = 0 \qquad \Rightarrow f'(0)=0$$

$$f'_+(x) = \lim_{x\to 0^+}\frac{-x^2+0^2}{x} = 0 ,$$

Above all

$$f'(x) = \begin{cases} 2x , & x>0 \\ 0 , & x=0 \\ -2x , & x<0 \end{cases}$$

(b) $f(x) = x^{\frac{1}{3}}(8-x)$

$$f'(x) = -\frac{4(x-2)}{3x^{\frac{2}{3}}}$$

$f'(x)=0 \Rightarrow x=2$

All critical points : $x=0$ and $x=2$

$f(0)=0$ $f(2)=6\sqrt[3]{2}$ $f(8)=0$

∴ Global Max : $6\sqrt[3]{2}$ Global Min : 0

Question 6

(a) $V = \frac{4}{3}\pi r^3$, $A = 4\pi r^2$

$$\frac{dV}{dt} = 4\pi r^2 \cdot \frac{dr}{dt}$$
$$= A\frac{dr}{dt}$$

Since its volume decreases at a rate proportional to its surface area $\Rightarrow \frac{dr}{dt}$ is a constant

When $V_t = \frac{1}{8}V_0$

$$\frac{4}{3}\pi (r)^3 = \frac{1}{8}\cdot\frac{4}{3}\pi (r_0)^3$$
$$r = \frac{1}{2}r_0$$

We know that $\frac{\Delta r}{\Delta t} = \frac{\frac{1}{2}r_0}{8\ hr} = \frac{1}{16}\ /hr$

When it melts completely

$t = 16$ hrs

(b) i) Assume $f(x) = x^2 - a$

$f'(x) = 2x$

According to Newton's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$= x_n - \frac{x_n^2 - a}{2x_n}$$
$$= x_n - \frac{x_n}{2} + \frac{a}{2x_n} = \frac{1}{2}(x_n + \frac{a}{x_n})$$

(ii) $a = 2$

$x_n = \frac{1}{2}x_n + \frac{1}{x_n}$

$x_0 = 1$

$x_1 = 1.5$

$x_2 \approx 1.417$

Question 7

(a) (i) $\int \tan^{-1}\frac{1}{x}dx$

$u=\tan^{-1}\frac{1}{x}$ $\quad dv=dx$

$du=-\frac{1}{x^2+1}\cdot dx$ $\quad v=x$

$$\int \tan^{-1}\frac{1}{x}\cdot dx = x\tan^{-1}\left(\frac{1}{x}\right)+\int\frac{x}{x^2+1}dx$$

$$= x\tan^{-1}\left(\frac{1}{x}\right)+\frac{1}{2}\ln(x^2+1)+C$$

(ii) $\int\frac{x}{x^2+6x+10}\cdot dx$

$$=\int\frac{2x+6}{2(x^2+6x+10)}\cdot dx-\int\frac{3}{x^2+6x+10}\cdot dx$$

$$=\frac{1}{2}\ln|x^2+6x+10|-\int\frac{3}{(x+3)^2+1}\cdot dx$$

$$=\frac{1}{2}\ln|x^2+6x+10|-3\tan^{-1}(x+3)+C$$

(b) (i) Assume $u=1-x \Rightarrow x=1-u$

then $du=-dx$

Left $=\int_1^0(1-u)^a u^b(-du)$

$$=-\int_1^0(1-u)^a u^b\cdot du$$

$$=\int_0^1 (1-u)^a u^b\cdot du$$

$$=\int_0^1(1-x)^a x^b\cdot dx$$ = Right

(ii) $\int_0^1 x(1-x)^{\frac{1}{3}}\cdot dx=\int_0^1 x^{\frac{1}{3}}(1-x)\cdot dx=\int_0^1x^{\frac{1}{3}}\cdot dx-\int_0^1x^{\frac{4}{3}}\cdot dx$

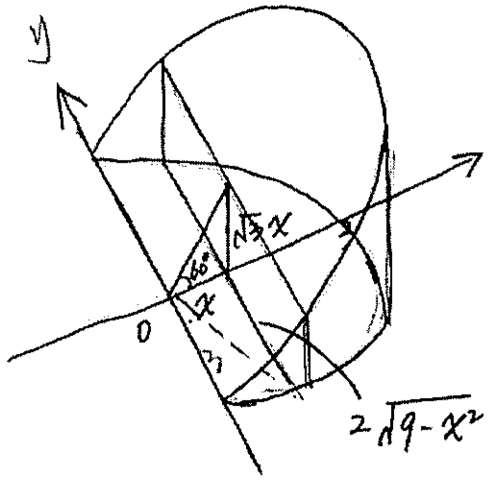
$$=\left[\frac{3}{4}x^{\frac{4}{3}}-\frac{3}{7}x^{\frac{7}{3}}\right]_0^1=\frac{3}{4}-\frac{3}{7}=\frac{9}{28}$$

(c)

$$A(x) = (\text{height})(\text{width})$$

$$= \sqrt{3}x \cdot 2\sqrt{9-x^2}$$

$$= 2\sqrt{3}x\sqrt{9-x^2}$$



$$V = \int_0^3 2\sqrt{3}\,x\sqrt{9-x^2}\cdot dx = \int_0^3 \sqrt{3}\cdot 2x\sqrt{9-x^2}\cdot dx$$

$$= -\frac{2}{3}\sqrt{3}\,(9-x^2)^{\frac{3}{2}}\Big]_0^3$$

$$= 18\sqrt{3}$$

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3 (10)	
4 (10)	

Questions	Marks
5 (15)	
6 (15)	
7 (25)	

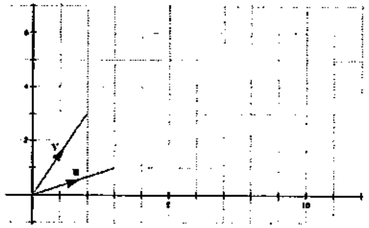
Total (100)	

QUESTION 1. **(12 Marks)**

(a) The diagram below shows two vectors **u** and **v**. Draw, on the diagram,

(i) the vector **u**+2**v**

(ii) the line $\ell : \mathbf{r} = \mathbf{v} + \lambda\mathbf{u}$, $\lambda \in \mathbb{R}$.



(b) Find the distance between the planes

$$\Pi_1 : \mathbf{r}\cdot(\mathbf{i}-2\mathbf{j}-2\mathbf{k}) = 3 \text{ and } \Pi_2 : 2x-4y-4z=0.$$

Question 1 continues on Page 3.

(c) Let r be a constant. Consider the following system of equations for variables x, y and z.

$$\begin{aligned} x^3 + z &= 1 \\ x^3 + y^5 &= 1 \\ y^5 + rz &= 1 \end{aligned}$$

(i) Find the values of r for which Cramer's rule is applicable.

(ii) For $r = 1$, use Cramer's Rule to find the unknown z.

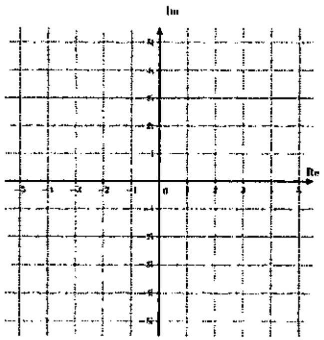
QUESTION 2. **(13 Marks)**

(a) Solve the equation $z^4 = -1024$. Express the roots in polar form $re^{i\theta}$, $r > 0$, $-\pi < \theta \leq \pi$.

(b) Express the roots found in part (a) in the form $x + iy$.

Question 2 continues on Page 5.

(c) Plot the four roots in the diagram below.



(d) Use your answers from part (b) to find the roots of the equation

$$\left(\frac{w}{2}+4i\right)^4=-1024.$$

Express your answers in the form $x+iy$.

QUESTION 3. **(10 Marks)**

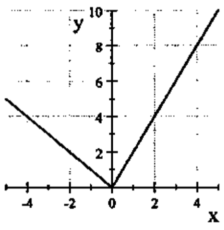
Evaluate the following limits.

(a) $\lim\limits_{x\to\infty}\dfrac{e^{2x}+1}{e^{3x}+1}\cos(x+1)$

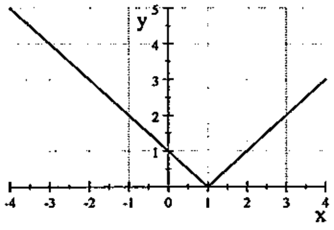
(b) $\lim\limits_{x\to1^-}\dfrac{\sqrt[3]{x}-1}{x-1}$

QUESTION 4. (10 Marks)

The graphs of piecewise linear functions f and g are given below.



Graph of $y = f(x)$



Graph of $y = g(x)$

(a) Find $\lim_{x \to -1} f(g(x))$.

(b) Find $(f \circ g)'(3)$.

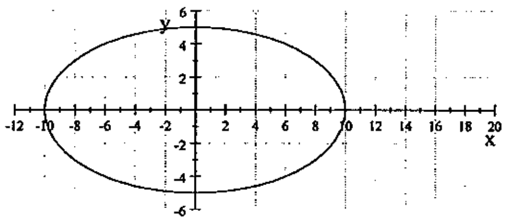
QUESTION 5 (15 Marks)

(a) (i) State the definition of the **derivative** of a function $f(x)$ at a point $x = a$.

(ii) Let $f(x) = x^3 \sin|x|$. Use the definition of derivative you have given in part (a)(i) to find the derivative of f at 0.

Question 5 continues on Page 9.

(b) (i) The graph of an ellipse with equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is given below, where a and b are positive integers. Find the values of a and b. Draw, on the graph below, the tangent line at the point $(6,4)$.



(ii) Show that the equation of the tangent line to the ellipse at the point (x_0, y_0) can be written as $\frac{x_0x}{a^2}+\frac{y_0y}{b^2}=1$.

QUESTION 6. **(15 Marks)**

(a) After a particular drug is taken, the concentration of the drug in the bloodstream is modeled by the function

$$c(t) = 27\left(e^{-0.4t} - e^{-0.6t}\right),$$

where the time t is measured in hours and the concentration c is measured in $\mu g/m\ell$. Find the maximum concentration of the drug during the first 6 hours after it is taken.

Question 6 continues on Page 11.

(b) Find the volume of the largest cylinder that can be inscribed in a sphere of radius r.

QUESTION 7. **(25 Marks)**

(a) Evaluate the following integrals.

(i) $\int \frac{1}{x^4 - 1}\, dx.$

(ii) $\int \frac{x}{\sqrt{6x - 8 - x^2}}\, dx.$

Question 7 continues on Page 13.

(b) (i) Use integration by parts to prove the reduction formula

$$\int \sin^n x \, dx = -\frac{1}{n} \cos x \sin^{n-1} x + \frac{n-1}{n} \int \sin^{n-2} x \, dx, \ n \geq 1.$$

(ii) Evaluate $\int_0^{\pi/2} \sin^5 x \, dx$.

Question 7 continues on Page 14.

(c) Let R be the region bounded by the curve of $y = e^{x^2}$, x-axis, y-axis and $x = 1$. Find the volume of the solid obtained by rotating R about the y-axis.

END OF PAPER

Appendix

Numerical Methods.

- Linearization Formula:

$$L(x) = f(a) + f'(a)\,(x-a)$$

- Newton's Method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

- Trapezoidal Rule:

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\left[y_0 + 2\,(y_1 + y_2 + \cdots + y_{n-1}) + y_n\right]$$

- Simpson's Rule:

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\left[y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n\right]$$

Derivatives.

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\sinh x) = \cosh x$$

$$\frac{d}{dx}(\tanh x) = \text{sech}^2 x$$

$$\frac{d}{dx}(\text{sech } x) = -\text{sech}x \tanh x$$

$$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{x^2+1}}$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(a^x) = a^x \ln a$$

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\cosh x) = \sinh x$$

$$\frac{d}{dx}(\coth x) = -\text{csch}^2 x$$

$$\frac{d}{dx}(\text{csch } x) = -\text{csch}x \coth x$$

Antiderivatives.

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C, \, n \neq -1 \qquad \int \frac{1}{x} \, dx = \ln|x| + C$$

$$\int \sin x \, dx = -\cos x + C \qquad \int \cos x \, dx = \sin x + C$$

$$\int \sec^2 x \, dx = \tan x + C \qquad \int \csc^2 x \, dx = -\cot x + C$$

$$\int \tan x \sec x \, dx = \sec x + C \qquad \int \cot x \csc x \, dx = -\csc x + C$$

$$\int \tan x \, dx = \ln|\sec x| + C \qquad \int \cot x \, dx = \ln|\sin x| + C$$

$$\int e^x \, dx = e^x + C \qquad \int a^x \, dx = \frac{a^x}{\ln a} + C, \, a > 0$$

$$\int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1} x + C \qquad \int \frac{1}{1+x^2} \, dx = \tan^{-1} x + C$$

$$\int \frac{1}{\sqrt{a^2-x^2}} \, dx = \sin^{-1}\left(\frac{x}{a}\right) + C, \, |x| < |a| \qquad \int \frac{1}{x^2+a^2} \, dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{\sqrt{x^2+1}} \, dx = \sinh^{-1} x + C \qquad \int \frac{1}{\sqrt{x^2+a^2}} \, dx = \sinh^{-1}\left(\frac{x}{a}\right) + C$$

MH1810 MATHEMATICS 1

Please read the following instructions carefully:

1. **Please do not turn over the question paper until you are told to do so. Disciplinary action may be taken against you if you do so.**

2. You are not allowed to leave the examination hall unless accompanied by an invigilator. You may raise your hand if you need to communicate with the invigilator.

3. Please write your Matriculation Number on the front of the answer book.

4. Please indicate clearly in the answer book (at the appropriate place) if you are continuing the answer to a question elsewhere in the book.

NANYANG TECHNOLOGICAL UNIVERSITY
SEMESTER I EXAMINATION 2016-2017
MH1810 – Mathematics 1

NOVEMBER 2016 TIME ALLOWED: 2 HOURS

Matriculation Number:

Seat Number:

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **SEVEN (7)** questions and comprises **SEVENTEEN (17)** pages, including an Appendix.
2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
3. This **IS NOT** an **OPEN BOOK** exam. However, a list of formulae is provided in the attachments.
4. Candidates may use calculators. However, they should write down systematically the steps in the workings.
5. All your solutions should be written in this booklet within the space provided after each question. If you use an additional answer book, attach it to this booklet and hand them in at the end of the examination.

For examiners only

Questions	Marks
1 (12)	
2 (13)	
3 (10)	
4 (10)	

Questions	Marks
5 (15)	
6 (15)	
7 (25)	

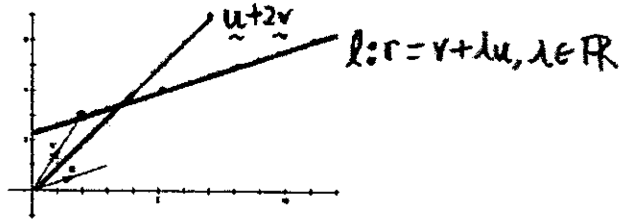
Total (100)	

QUESTION 1. (12 Marks)

(a) The diagram below shows two vectors **u** and **v**. Draw, on the diagram,

(i) the vector **u**+2**v**

(ii) the line $\ell : \mathbf{r} = \mathbf{v} + \lambda\mathbf{u},\ \lambda \in \mathbb{R}$.



(b) Find the distance between the planes

$$\Pi_1 : \mathbf{r}\cdot(\mathbf{i}-2\mathbf{j}-2\mathbf{k}) = 3 \text{ and } \Pi_2 : 2x - 4y - 4z = 0.$$

Question 1 continues on Page 3.

(c) Let r be a constant. Consider the following system of equations for variables x, y and z.

$$\begin{aligned} x^3 &+ z = 1 \\ x^3 &+ y^5 = 1 \\ y^5 &+ rz = 1 \end{aligned}$$

(i) Find the values of r for which Cramer's rule is applicable.

(ii) For $r = 1$, use Cramer's Rule to find the unknown z.

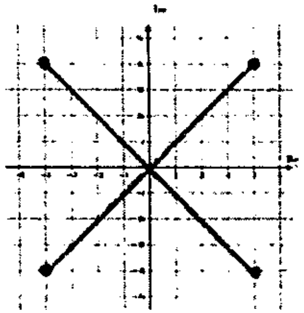
QUESTION 2. **(13 Marks)**

(a) Solve the equation $z^4 = -1024$. Express the roots in polar form $re^{i\theta}$, $r > 0$, $-\pi < \theta \leq \pi$.

(b) Express the roots found in part (a) in the form $x + iy$.

Question 2 continues on Page 5.

(c) Plot the four roots in the diagram below.



(d) Use your answers from part (b) to find the roots of the equation

$$\left(\frac{w}{2}+4i\right)^4=-1024.$$

Express your answers in the form $x+iy$.

QUESTION 3. **(10 Marks)**

Evaluate the following limits.

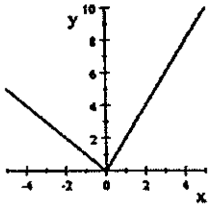
(a) $\lim_{x\to\infty}\frac{e^{2x}+1}{e^{3x}+1}\cos(x+1)$

(b) $\lim_{x\to1^-}\frac{\sqrt[3]{x}-1}{x-1}$

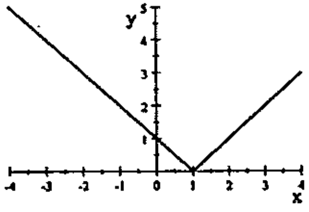
QUESTION 4.

(10 Marks)

The graphs of piecewise linear functions f and g are given below.



Graph of $y = f(x)$



Graph of $y = g(x)$

(a) Find $\lim_{x \to -1} f(g(x))$.

(b) Find $(f \circ g)'(3)$.

QUESTION 5

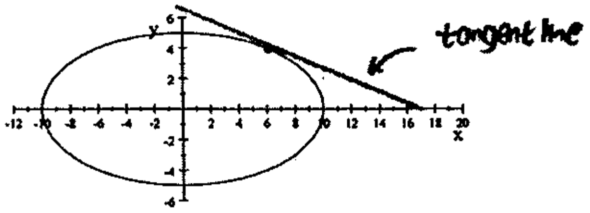
(15 Marks)

(a) (i) State the definition of the derivative of a function $f(x)$ at a point $x = a$.

(ii) Let $f(x) = x^3 \sin|x|$. Use the definition of derivative you have given in part (a)(i) to find the derivative of f at 0.

Question 5 continues on Page 9.

(b) (i) The graph of an ellipse with equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is given below, where a and b are positive integers. Find the values of a and b. Draw, on the graph below, the tangent line at the point $(6,4)$.



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END OF PAPER

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Subject Name	: Mathematics 1
Year / Semester :	2016–2017 | SEM 1

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①b $\pi_1 : r \cdot \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix} = 3$

$\pi_2 : 2x - 4y - 4z = 0 \xrightarrow{\text{can be written as}} x - 2y - 2z = 0$

$r \cdot \begin{pmatrix} 2 \\ -4 \\ -4 \end{pmatrix} = 0 \qquad r \cdot \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix} = 0$

method 1

Since $\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix} = 3(1) + 0(-2) + 0(-2)$

$= 3,$

$\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$ lies on π_1

Since $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -4 \\ -4 \end{pmatrix} = 0(2) + 0(-4) + 0(-4)$

$= 0,$

$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ lies on π_2

$\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$

$$\text{distance} = \frac{\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}}{\sqrt{1^2 + (-2)^2 + (-2)^2}}$$

$$= \frac{3}{\sqrt{9}}$$

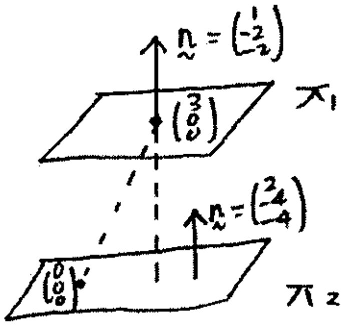
$$= 1$$

method 2

$\pi_1 \,//\, \pi_2$

$$\text{distance} = \frac{3 - 0}{\sqrt{1^2 + (-2)^2 + (-2)^2}}$$

$$= \frac{3}{\sqrt{9}} = 1$$



1ci)

$x^3+z=1$

$x^3+y^5=1$

$y^5+rz=1$

vector $\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & r \end{pmatrix}\begin{pmatrix} x^3 \\ y^5 \\ z \end{pmatrix}=\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

For Cramer's Rule to be applicable, $\det\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & r \end{pmatrix} \neq 0$

$\det\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & r \end{vmatrix} = 1\cdot\det\begin{vmatrix} 1 & 0 \\ 1 & r \end{vmatrix} + 0\cdot\det\begin{vmatrix} 1 & 0 \\ 0 & r \end{vmatrix} + 1\cdot\det\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}$

Taking 1st row $= (r-0)+(1-0)$

$= r+1 \neq 0$

$r \neq -1$

ii) $\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} x^3 \\ y^5 \\ z \end{pmatrix}=\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$\det\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = 1+1$

$= 2$

By Cramer's Rule,

$z = \dfrac{\det\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}}{\det\left|\begin{matrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{matrix}\right)}$

$= \dfrac{\det\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + \det\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}}{2}$ Taking 1st row

$= \dfrac{1}{2}$

Could Use GC to double check the answer

2a) $z^4 = -1024$

$= 2^{10} e^{i(\pi)}$

$= 2^{10} e^{i(\pi + 2k\pi)}, \quad k = 0, \pm 1 \pm 2$

$z = 2^{\frac{10}{4}} e^{i\left(\frac{\pi + 2k\pi}{4}\right)}, \quad k = 0, \pm 1, \pm 2$

$z = \underbrace{2^{\frac{5}{2}} e^{i(\frac{\pi}{4})}, 2^{\frac{5}{2}} e^{i(-\frac{\pi}{4})}}, \underbrace{2^{\frac{5}{2}} e^{i(\frac{3\pi}{4})}, 2^{\frac{5}{2}} e^{i(-\frac{3\pi}{4})}}$

conjugate pairs

$\sin(-\theta) = -\sin\theta$

$\cos(-\theta) = \cos\theta$

b) $z = 2^{\frac{5}{2}}\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right], 2^{\frac{5}{2}}\left[\cos(-\frac{\pi}{4}) + i\sin(-\frac{\pi}{4})\right], 2^{\frac{5}{2}}\left[\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right], 2^{\frac{5}{2}}\left[\cos(-\frac{3\pi}{4}) + i\sin(-\frac{3\pi}{4})\right]$

$= 2^{\frac{5}{2}}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i\right), 2^{\frac{5}{2}}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i\right), 2^{\frac{5}{2}}\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i\right), 2^{\frac{5}{2}}\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i\right)$

$= 4 + 4i, 4 - 4i, -4 + 4i, -4 - 4i$

Note: Double check the answers with GC

d) $\left(\frac{W}{2} + 4i\right)^4 = -1024$

Let $z = \frac{W}{2} + 4i$, $z^4 = -1024$ which is same as part (a) & (b)

$\frac{W}{2} + 4i = 4 + 4i, 4 - 4i, -4 + 4i, -4 - 4i$

$\frac{W}{2} = 4, \quad 4 - 8i, \quad -4, \quad -4 - 8i$

$W = 8, \quad 8 - 16i, \quad -8, \quad -8 - 16i$

3a) $\lim\limits_{x\to\infty} \frac{e^{2x}+1}{e^{3x}+1} \cos(x+1) = \lim\limits_{x\to\infty} \frac{e^{2x}+1}{e^{3x}+1} \cdot \lim\limits_{x\to\infty} \cos(x+1)$

$-1 \leqslant \cos(x+1) \leq 1 \rightarrow \lim\limits_{x\to\infty} \cos(x+1) = C$, constant, where $-1 \leqslant C \leq 1$

$$\lim_{x\to\infty} \frac{e^{2x}+1}{e^{3x}+1} \cdot \lim_{x\to\infty} \cos(x+1) = C \cdot \lim_{x\to\infty} \frac{e^{2x}+1}{e^{3x}+1} \quad \left(\frac{\infty}{\infty}\right)$$

$$= C \lim_{x\to\infty} \frac{2e^{2x}}{3e^{3x}}$$ with L' Hospital Rule

$$= C \lim_{x\to\infty} \frac{2}{3e^{x}}$$

$$= C \cdot 0 = 0$$

b) $\lim\limits_{x\to 1^-} \frac{\sqrt[3]{x}-1}{x-1} = \lim\limits_{y\to 1^-} \frac{y-1}{y^3-1}$

Let $y=\sqrt[3]{x}$,

$$= \lim_{y\to 1^-} \frac{1}{y^2+y+1}$$

$$= \frac{1}{3}$$

$$\begin{array}{r|l} & y^2+y+1 \\ \hline y-1 & y^3 \quad\quad -1 \\ & y^3-y^2 \\ \hline & y^2 \quad -1 \\ & y^2-y \\ \hline & y-1 \\ & y-1 \\ \hline & 0 \end{array}$$

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4a) $\lim_{x\to -1} f(g(x)) = \lim_{x\to -1} f(2) = 4$

Sub $x=-1$ for $y=g(x)$ to find the value of y,
then use the value of y to sub in $y'=f(y)$,
y' will be the final answer

b) $(f \circ g)'(x) = f' g(x) \cdot g'(x)$ chain Rule

$(f \circ g)'(3) = f' g(3) \cdot g'(3)$

$= 2 \cdot 1$

$= 2$

$g'(3) = \frac{3-0}{4-1}$

$=1$

5ai) $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$x=0$ $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(0)}{h}$

ii) $f(x) = x^3 \sin|x|$

$$= \begin{cases} x^3 \sin x & , x>0 \\ x^3 \sin(-x) = -x^3 \sin x & , x<0 \end{cases}$$

To find $f'(0)$, We need to see LHS & RHS

$$f'(0^+) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{h^3 \sin h - 0}{h}$$

$$= \lim_{h \to 0} h^2 \sin h$$

$$= 0$$

$$f'(0^-) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{-h^3 \sin h - 0}{h}$$

$$= \lim_{h \to 0} -h^2 \sin h$$

$$= 0$$

$$f'(0^+) = f'(0^-) = 0$$

$$f'(0) = 0$$

5bii)

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Differentiating wrt x,

$$\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$$

$$\frac{2y}{b^2}\frac{dy}{dx}=-\frac{2x}{a^2}$$

$$\frac{dy}{dx}=-\frac{xb^2}{ya^2}$$

$$\left.\frac{dy}{dx}\right|_{x=x_0}=-\frac{x_0b^2}{y_0a^2}$$

eq$\underline{\text{n}}$ of tangent line $\quad y-y_0=-\frac{x_0b^2}{y_0a^2}(x-x_0)$

$$yy_0a^2-y_0^2a^2=-xx_0b^2+x_0^2b^2$$

$$yy_0a^2+xx_0b^2=(y_0a)^2+(x_0b)^2 \quad -①$$

From eq$\underline{\text{n}}$ $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, If we sub $x=x_0$, y will become y_0

$$\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}=1$$

$$(x_0b)^2+(y_0a)^2=(ab)^2 \quad -②$$

sub ② in ① $\quad yy_0a^2+xx_0b^2=a^2b^2$

$$\frac{x_0x}{a^2}+\frac{y_0y}{b^2}=1 \quad \text{(shown)}$$

6a) $C(t) = 27(e^{-0.4t} - e^{-0.6t})$

$C'(t) = 27(-0.4e^{-0.4t} + 0.6e^{-0.6t})$

For stationary point(s), $C'(t) = 0$

$27(-0.4e^{-0.4t} + 0.6e^{-0.6t}) = 0$

$0.4e^{-0.4t} = 0.6e^{-0.6t}$

$\frac{e^{-0.4t}}{e^{-0.6t}} = \frac{0.6}{0.4}$

$e^{0.2t} = 1.5$

$\ln e^{0.2t} = \ln 1.5$

$0.2t = \ln 1.5$

$t = 5\ln 1.5$

[OR] $C''(t) = 27(0.16e^{-0.4t} - 0.36e^{-0.6t})$

$C''(5\ln 1.5) = 27(0.16e^{-2\ln 1.5} - 0.36e^{-3\ln 1.5})$

$= 27\left(-\frac{8}{225}\right) < 0$

∴ maximum point

t	−	5ln1·5	+
C'(t)	/	—	\

∴ maximum point

extreme points (0,0) , (6, 1.71)

stationary point $C(5\ln 1.5) = 27(e^{-2\ln 1.5} - e^{-3\ln 1.5})$

$= 4 \ \mu g/ml$

maximum concentration of the drug = $4\mu g/ml$

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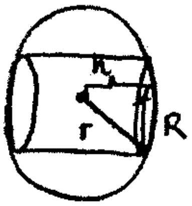
6b)



let Radius of cylinder be R

$h^2+R^2=r^2$

$h=\sqrt{r^2-R^2}$

Volume of cylinder, V = base area x height

$= \pi R^2 h$

$= \pi R^2\sqrt{r^2-R^2}$

$$\frac{dV}{dR} = 2\pi R\sqrt{r^2-R^2} + \frac{\pi R^2}{\cancel{2}\sqrt{r^2-R^2}}(-\cancel{2}R)$$

$$= \frac{2\pi R(r^2-R^2)-\pi R^3}{\sqrt{r^2-R^2}}$$

$\frac{dV}{dR}=0$ for stationary point(s)

$2\pi Rr^2 - 3\pi R^3 = 0$

$\pi R(2r^2-3R^2)=0$

$R=0$ OR $2r^2-3R^2=0$

$R=\sqrt{\frac{2r^2}{3}}$

R	$-$	$\sqrt{\frac{2r^2}{3}}$	$+$
$\frac{dV}{dR}$	/	—	\

∴ maximum pt

$V=\pi R^2 h$

$= \pi\frac{2r^2}{3}\sqrt{r^2-\frac{2r^2}{3}}$

$= \frac{2\pi r^3}{3\sqrt{3}}$

[OR] $$\frac{d^2V}{dR^2} = \frac{(2\pi r^2-9\pi R^2)}{\sqrt{r^2-R^2}} - \frac{(2\pi Rr^2-3\pi R^3)\left(\frac{1}{\cancel{2}}\right)(-\cancel{2}R)}{\sqrt{(r^2-R^2)^3}}$$

$$\left.\frac{d^2V}{dR^2}\right|_{R=\sqrt{\frac{2r^2}{3}}} = \frac{2\pi r^2-9\pi\left(\frac{2}{3}r^2\right)}{\sqrt{r^2-\frac{2r^2}{3}}} - \frac{2\pi\left(\frac{2}{3}r^2\right)r^2-3\pi\left(\frac{2r^2}{3}\right)}{\sqrt{\left(r^2-\frac{2r^2}{3}\right)^3}}$$

$$= \frac{-4\pi r^2}{\sqrt{\frac{r^2}{3}}} - \frac{\frac{4}{3}\pi r^4-\frac{4}{3}\pi r^4}{\sqrt{\left(r^2-\frac{2r^2}{3}\right)^3}}$$

$$= -\frac{4\pi r^2}{\sqrt{\frac{r^2}{3}}} < 0$$ ∴ maximum point

7a i) $\int \frac{1}{x^4-1}\,dx = \int \frac{1}{(x^2+1)(x^2-1)}\,dx = \int \frac{1}{(x^2+1)(x+1)(x-1)}\,dx$

Note: $y^2-x^2=(y-x)(y+x)$

$$\frac{1}{(x^2+1)(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{Cx+D}{x^2+1}$$

$$1 = A(x-1)(x^2+1)+B(x+1)(x^2+1)+(Cx+D)(x+1)(x-1)$$

sub $x=1$ $\quad 1 = B(2)(2)$

$B=\frac{1}{4}$

sub $x=-1$ $\quad 1 = A(-2)(2)$

$A=-\frac{1}{4}$

sub $x=0$ $\quad 1 = -\frac{1}{4}(-1)(1)+\frac{1}{4}(1)(1)+D(1)(-1)$

$=\frac{1}{2}-D$

$D=-\frac{1}{2}$

sub $x=2$ $\quad 1 = -\frac{1}{4}(1)(5)+\frac{1}{4}(3)(5)+(2C-\frac{1}{2})(3)(1)$

$=\frac{5}{2}+3(2C-\frac{1}{2})$

$-\frac{3}{2}=3(2C-\frac{1}{2})$

$2C-\frac{1}{2}=-\frac{1}{2}$

$2C=0$

$C=0$

$$\int \frac{1}{x^4-1}\,dx = \int -\frac{1}{4(x+1)} + \frac{1}{4(x-1)} - \frac{1}{2(x^2+1)}\,dx$$

$$= -\frac{1}{4}\ln|x+1| + \frac{1}{4}\ln|x-1| - \frac{1}{2}\tan^{-1}x + C$$

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$$\frac{d}{dx}(6x-8-x^2)=-2x+6$$
$$6x-8-x^2=1-(x-3)^2$$

7aiii) $$\int \frac{x}{\sqrt{6x-8-x^2}}\,dx = -\frac{1}{2}\int \frac{-2x+6-6}{\sqrt{6x-8-x^2}}\,dx$$
$$= -\frac{1}{2}\int \frac{6-2x}{\sqrt{6x-8-x^2}} - \frac{6}{\sqrt{6x-8-x^2}}\,dx$$
$$= \frac{1}{2}\int \frac{6}{\sqrt{1-(x-3)^2}} - \frac{6-2x}{\sqrt{6x-8-x^2}}\,dx$$
$$= \frac{1}{2}\left[6\sin^{-1}\left(\frac{x-3}{1}\right) - 2\sqrt{6x-8-x^2} + C\right]$$
$$= 3\sin^{-1}(x-3) - \sqrt{6x-8-x^2} + C$$

bi) $$\int \sin^n x\,dx = \int \sin^{n-1}x \sin x\,dx$$
$$= -\cos x \sin^{n-1}x - \int (-\cos x)(n-1)\sin^{n-2}x \cos x\,dx$$
$$= -\cos x \sin^{n-1}x + (n-1)\int \cos^2 x \sin^{n-2}x\,dx$$
$$= -\cos x \sin^{n-1}x + (n-1)\int (1-\sin^2 x)\sin^{n-2}x\,dx$$
$$= -\cos x \sin^{n-1}x + (n-1)\int \sin^{n-2}x - \sin^n x\,dx$$
$$= -\cos x \sin^{n-1}x + (n-1)\int \sin^{n-2}x\,dx - \underbrace{(n-1)\int \sin^n x\,dx}_{\text{Bring to LHS}}$$

$$(n-1+1)\int \sin^n x\,dx = -\cos x \sin^{n-1}x + (n-1)\int \sin^{n-2}x\,dx$$
$$n\int \sin^n x\,dx = -\cos x \sin^{n-1}x + (n-1)\int \sin^{n-2}x\,dx$$
$$\int \sin^n x\,dx = -\frac{1}{n}\cos x \sin^{n-1}x + \frac{n-1}{n}\int \sin^{n-2}x\,dx$$ (shown)

7bii) $\int_0^{\frac{\pi}{2}} \sin^5 x\,dx = \left[-\frac{1}{5}\cos x\sin^4 x\right]\Big|_0^{\frac{\pi}{2}} + \frac{4}{5}\int_0^{\frac{\pi}{2}}\sin^3 x\,dx$

$= \left(-\frac{1}{5}\cos\frac{\pi}{2}\sin^4\frac{\pi}{2} + \frac{1}{5}\cos 0\sin^4 0\right) + \frac{4}{5}\left[-\frac{1}{3}\cos x\sin^2 x\right]\Big|_0^{\frac{\pi}{2}} + \frac{4}{5}\left(\frac{2}{3}\right)\int_0^{\frac{\pi}{2}}\sin x\,dx$

$= \frac{4}{5}\left(\frac{2}{3}\right)\int_0^{\frac{\pi}{2}}\sin x\,dx$

$= \frac{8}{15}\left[-\cos x\right]\Big|_0^{\frac{\pi}{2}}$

$= \frac{8}{15}\left[-\cos\frac{\pi}{2} - (-\cos 0)\right]$

$= \frac{8}{15}$ Use GC to double check the answer

7c) Volume $= 2\pi\int_0^1 xy\,dx$

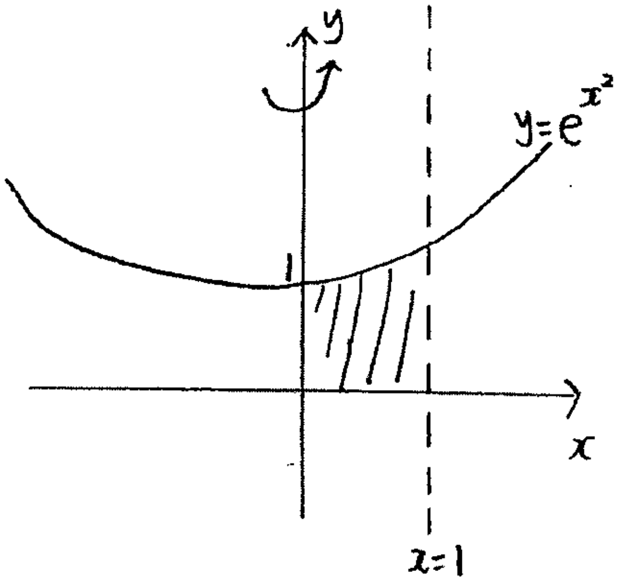
$= 2\pi\int_0^1 xe^{x^2}\,dx$

$= \pi\int_0^1 2xe^{x^2}\,dx$

$= \pi\left[e^{2x}\right]\Big|_0^1$

$= \pi(e^2 - e^0)$

$= \pi(e^2 - 1)$



NANYANG TECHNOLOGICAL UNIVERSITY
SEMESTER I EXAMINATION 2017–2018
MH1810 – Mathematics 1

NOVEMBER 2017 TIME ALLOWED: 2 HOURS

Matriculation Number:

Seat Number:

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **SEVEN (7)** questions and comprises **NINETEEN (19)** pages, including an Appendix.
2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
3. This **IS NOT** an **OPEN BOOK** exam. However, a list of formulae is provided in the attachments.
4. Candidates may use calculators. However, they should write down systematically the steps in the workings.
5. All your solutions should be written in this booklet within the space provided after each question. If you use an additional answer book, attach it to this booklet and hand them in at the end of the examination.

For examiners only

Questions	Marks
1 (10)	
2 (10)	
3 (10)	
4 (15)	

Questions	Marks
5 (15)	
6 (15)	
7 (25)	

Total (100)	

QUESTION 1. **(10 Marks)**

(a) In Figure 1, AB is the diameter of a circle and C is a point on the arc joining A and B. Let $\mathbf{u} = \overrightarrow{OC}$ and $\mathbf{v} = \overrightarrow{OB}$, where O is the center of the circle.

(i) Draw, on the diagram, the vectors $\mathbf{v} - \mathbf{u}$ and $-\mathbf{u} - \mathbf{v}$, with C as the initial point (starting point).

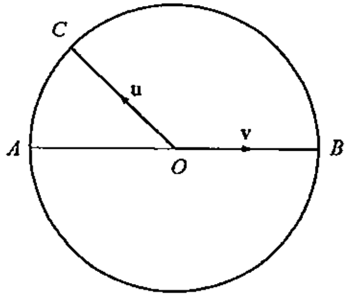


Figure 1

(ii) Use dot product to show that $\overrightarrow{CA}$ is perpendicular to $\overrightarrow{CB}$.

Question 1 continues on Page 3.

(b) Find the area of triangle whose vertices are $A\,(1,0,0)$, $B\,(0,1,0)$, and $C\,(0,0,1)$. Hence, deduce the distance from A to the line BC.

QUESTION 2. **(10 Marks)**

(a) Let $z = a + ai$, where a is a *negative* real number.

  (i) Find the modulus and principal argument of z.

  (ii) Show that z^{16} is a real number. What is that number?

Question 2 continues on Page 5.

(b) Let $z = -1 - i$. Plot the points z, z^3, z^5, z^7 on the Argand diagram below (Figure 2).

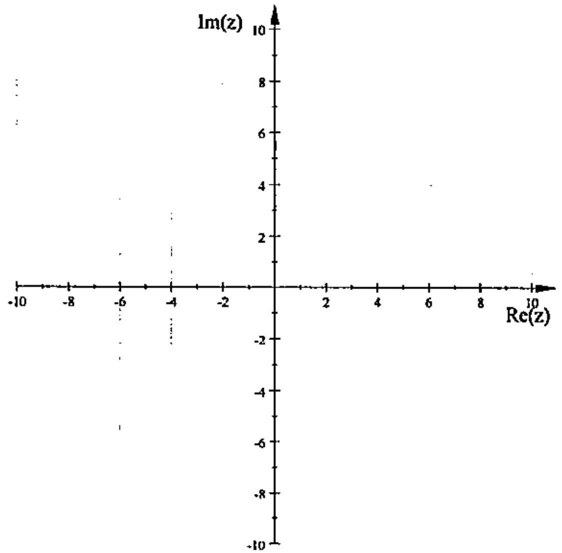


Figure 2

QUESTION 3. **(10 Marks)**

Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ x & 0 & 1 & x \\ 1 & 0 & x & 0 \\ x & 0 & 0 & 1 \end{bmatrix}, \text{ where } x \text{ is a real number.}$$

(a) Show that determinant of A is $x^3 - x^2 + 1$.

Question 3 continues on Page 7.

(b) Show that A is singular for some value of $x \in (-1,0)$. State the theorem used.

QUESTION 4. **(15 Marks)**

(a) Figure 3 shows the graph of a differentiable function $f(x)$. Given that $(1,-2)$ and $(-1,2)$ are the local minimum and maximum of the graph, and the graph has a minimum gradient of -3 at $x=0$, **sketch the graph of $f'(x)$ on the same diagram** (Figure 3).

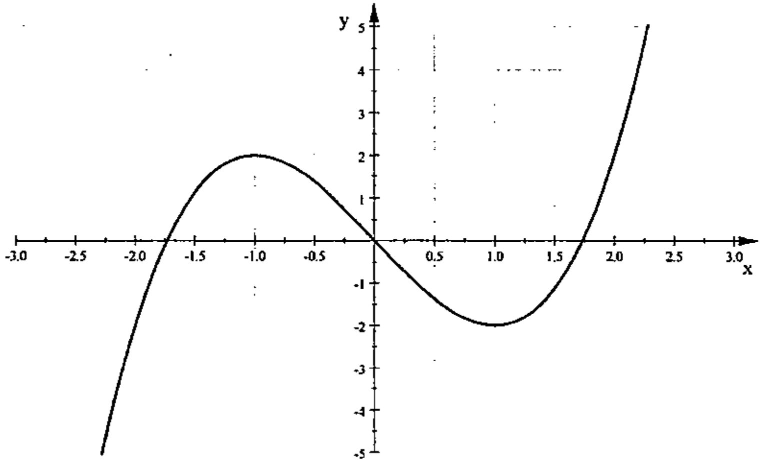


Figure 3

Question 4 continues on Page 9.

(b) Figure 4 is the graph of a piecewise linear function $f(x)$ (i.e., the graph of $y = f(x)$ is made up of straight lines).

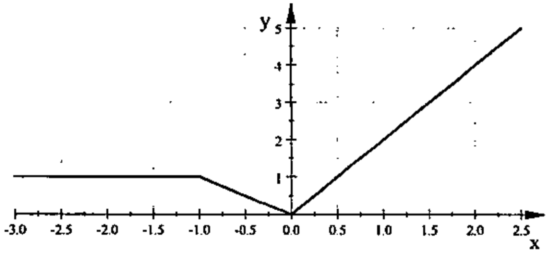


Figure 4

(i) State the definition of the derivative $f'(c)$ of $f(x)$ at $x = c$. Use **the definition of derivative** to show that $f'(0)$ does not exist.

(ii) Evaluate $\int_{-2}^{2} f(x)\, dx$.

QUESTION 5 **(15 Marks)**

(a)(i) State, without proof, the Mean Value Theorem.

(ii) If $f'(x) > 0$ for all $x \in \mathbb{R}$, use **the Mean Value Theorem** to show that f is an increasing function.

Question 5 continues on Page 11.

(b) Compute the following limits.

(i) $\lim\limits_{x\to-\infty} \dfrac{x}{\sqrt{x^2+1}}$

(ii) $\lim\limits_{x\to1} \left(\dfrac{1}{\ln x} - \dfrac{x}{x-1}\right)$

QUESTION 6. **(15 Marks)**

(a) A street light is mounted at the top of a 6-metre-tall pole. A man 2 m tall is walking away from the pole at a speed of 2 m/s along a straight path. How fast is the tip of his shadow moving when he is 10 m away from the pole?

Question 6 continues on Page 13.

(b) Find the global maximum and minimum of the function $f(x) = \ln(x^2+1)+x$ on the interval $[-2, 2]$.

QUESTION 7. **(25 Marks)**

(a) Evaluate the following integrals.

(i) $\int \frac{x+2}{x^2+5x-6}\,dx$,

(ii) $\int x\tan^{-1} x dx$.

Question 7 continues on Page 15.

(b) Find the values of p for which the improper integral $\int_e^{\infty} \frac{1}{x(\ln x)^p} dx$ converges and evaluate the integrals for those values of p.

Question 7 continues on Page 16.

(c) Use Simpson's Rule and Trapezoidal Rule with $n = 10$ to approximate $\int_0^1 \frac{1}{x+1} dx$.

Show that S_{10} is a better approximation to the actual value of $\int_0^1 \frac{1}{x+1} dx$ than T_{10}.

END OF PAPER

Appendix

Numerical Methods.

- Linearization Formula:

$$L(x) = f(a) + f'(a)\,(x - a)$$

- Newton's Method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

- Trapezoidal Rule:

$$\int_a^b f(x)\,dx \approx T_n = \frac{h}{2}\,[y_0 + 2\,(y_1 + y_2 + \cdots + y_{n-1}) + y_n]$$

- Simpson's Rule:

$$\int_a^b f(x)\,dx \quad\approx\quad S_n = \frac{h}{3}\,[y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n]\,,$$

where n is even.

Derivatives.

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(e^x) = e^x \qquad \frac{d}{dx}(a^x) = a^x \ln a$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x} \qquad \frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}} \qquad \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1 + x^2}$$

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\cosh x) = \sinh x$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \qquad \frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$

$$\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x \qquad \frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

$$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{x^2 + 1}}$$

Antiderivatives.

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1 \qquad \int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \csc^2 x\,dx = -\cot x + C$$

$$\int \tan x \sec x\,dx = \sec x + C \qquad \int \cot x \csc x\,dx = -\csc x + C$$

$$\int \tan x\,dx = \ln|\sec x| + C \qquad \int \cot x\,dx = \ln|\sin x| + C$$

$$\int e^x\,dx = e^x + C \qquad \int a^x\,dx = \frac{a^x}{\ln a} + C,\ a > 0$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C \qquad \int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C,\ |x| < |a| \qquad \int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{\sqrt{x^2+1}}\,dx = \sinh^{-1} x + C \qquad \int \frac{1}{\sqrt{x^2+a^2}}\,dx = \sinh^{-1}\left(\frac{x}{a}\right) + C$$

Subject Code : MH1810

Subject Name : Mathematics I

Year / Semester : AY17-18 / Semester 1

Suggested Solution Brought to You by Materials Science and Engineering Club

1. a. In Figure 1, AB is the diameter of a circle and C is a point on the arc joining A and B. Let $\mathbf{u} = \overrightarrow{OC}$ and $\mathbf{v} = \overrightarrow{OB}$, where O is the center of the circle.

i) Draw, on the diagram, the vectors **v-u** and **-u-v**, with C as the initial point (starting point).

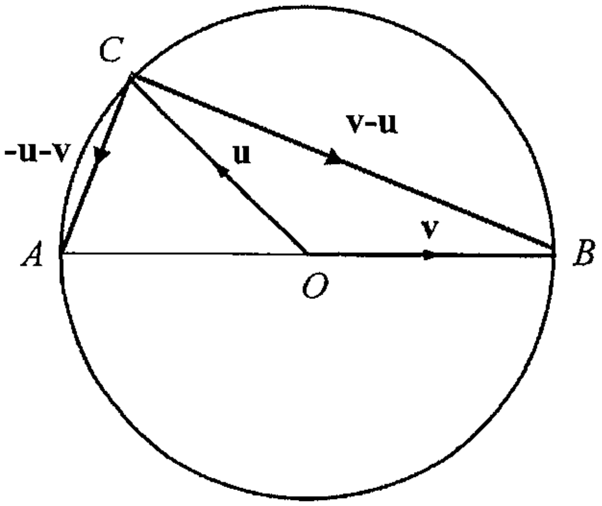


Figure 1

ii) Use dot product to show that $\overrightarrow{CA}$ is perpendicular to $\overrightarrow{CB}$.

$$\text{Show } \overrightarrow{CA} \,.\, \overrightarrow{CB} = 0$$
$$(-\mathbf{u}-\mathbf{v}) \,.\, (\mathbf{v}-\mathbf{u}) = 0$$
$$\cancel{-\mathbf{u}.\mathbf{v}} + \mathbf{u}.\mathbf{u} - \mathbf{v}.\mathbf{v} + \cancel{\mathbf{u}.\mathbf{v}} = 0$$
$$\mathbf{u}.\mathbf{u} - \mathbf{v}.\mathbf{v} = 0$$
$$||\mathbf{u}||^2 - ||\mathbf{v}||^2 = 0$$
$$||\mathbf{u}||^2 - ||\mathbf{u}||^2 = 0$$
$$0 = 0$$

<shown>

$\therefore \overrightarrow{CA}$ is perpendicular to $\overrightarrow{CB}$.

1. b. Find the area of triangle whose vertices are A (1,0,0), B (0,1,0), and C (0,0,1). **Hence**, deduce the distance from A to the line BC.

$$\overrightarrow{AB} = \vec{B} - \vec{A} = \begin{pmatrix}0\\1\\0\end{pmatrix} - \begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}-1\\1\\0\end{pmatrix}$$

$$\overrightarrow{AC} = \vec{C} - \vec{A} = \begin{pmatrix}0\\0\\1\end{pmatrix} - \begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}-1\\0\\1\end{pmatrix}$$

$$\begin{aligned}\text{Area of Triangle} &= \frac{1}{2}\left\|\overrightarrow{AB}\ x\ \overrightarrow{AC}\right\| \\ &= \frac{1}{2}\left\|\begin{pmatrix}-1\\1\\0\end{pmatrix} \text{x} \begin{pmatrix}-1\\0\\1\end{pmatrix}\right\| \\ &= \frac{1}{2}\left\|\begin{vmatrix}1 & 0\\0 & 1\end{vmatrix}, -\begin{vmatrix}-1 & -1\\0 & 1\end{vmatrix}, \begin{vmatrix}-1 & -1\\1 & 0\end{vmatrix}\right\| \\ &= \frac{1}{2}\left\|\begin{pmatrix}1\\1\\1\end{pmatrix}\right\| \\ &= \frac{1}{2}\sqrt{1^2 + 1^2 + 1^2} \\ &= \frac{\sqrt{3}}{2}\ \text{units}^2\end{aligned}$$

∴ The area of the triangle is $\frac{\sqrt{3}}{2}$ units^2.

→ The distance from A to line BC is the **height of the triangle ABC with base BC.**

$$\overrightarrow{BC} = \vec{C} - \vec{B} = \begin{pmatrix}0\\0\\1\end{pmatrix} - \begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\-1\\1\end{pmatrix}$$

$$\left\|\overrightarrow{BC}\right\| = \sqrt{0^2 + -1^2 + 1^2} = \sqrt{2}$$

$$\begin{aligned}\text{Area} &= \text{Base x Height} \\ \frac{\sqrt{3}}{2} &= \sqrt{2}\ \text{x Height} \\ \text{Height} &= \frac{\sqrt{6}}{4}\ \textbf{units}\end{aligned}$$

∴ The distance from A to line BC is $\frac{\sqrt{6}}{4}$ units.

2\. a. Let $z = a + ai$, where a is a negative real number.

i) Find the modulus and principal argument of z.

Modulus of z:

$$|z| = \sqrt{a^2 + a^2}$$
$$= \sqrt{2a^2}$$
$$|z| = -\sqrt{2}\boldsymbol{a}$$

Argument of z:

\- z is located at the third quadrant.

$$\theta = \tan^{-1}\left(\frac{a}{a}\right)$$
$$= \tan^{-1}(1)$$
$$\boldsymbol{\theta = -\frac{3\pi}{4}}$$

ii) Show that z^{16} is a real number. What is that number?

$$z = a + ai$$
$$z = re^{i\theta} \,||\, z^n = r^n e^{in\theta}$$
$$z = (-\sqrt{2}a)e^{i\left(-\frac{3\pi}{4}\right)}$$
$$z^{16} = (-\sqrt{2}a)^{16} e^{i\left(-\frac{3\pi}{4}\right)\cdot 16}$$
$$z^{16} = 256a^{16}e^{i(-12\pi)}$$
$$z^{16} = 256a^{16}e^{i0}$$
$$z^{16} = 256a^{16}$$

a is a negative real number, therefore z^{16} is a real number. <*shown*>

∴ **That number is $\underline{256a^{16}}$.**

2. b. Let $z = -1 - i$. Plot the points z, z^3, z^5, z^7 on the Argand diagram below (Figure 2).

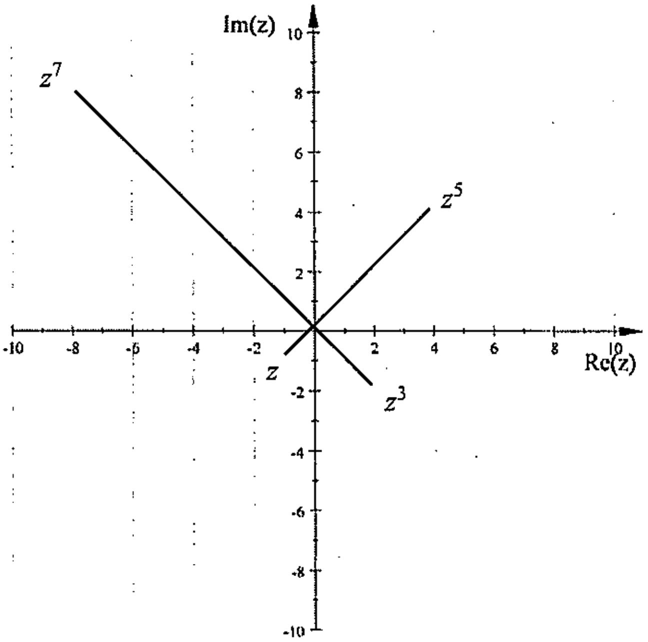


Figure 2

$z = -1 - i$

Modulus of z:

$$|z| = \sqrt{-1^2 + -1^2}$$
$$|z| = \sqrt{2}$$

Argument of z:

- z is located at the third quadrant.

$$\theta = \tan^{-1}\left(\frac{1}{1}\right)$$
$$= \tan^{-1}(1)$$
$$\theta = -\frac{3\pi}{4}$$

$$z = re^{i\theta} \,\|\, z^n = r^n e^{in\theta}$$
$$z = \sqrt{2}e^{i\left(-\frac{3\pi}{4}\right)}$$

$$z^3 = (\sqrt{2})^3 e^{i\left(-\frac{3\pi}{4}\right)\cdot 3}$$
$$z^3 = 2\sqrt{2}e^{i\left(-\frac{9\pi}{4}\right)}$$
$$z^3 = 2\sqrt{2}e^{i\left(-\frac{\pi}{4}\right)} \rightarrow 4^{\text{th}} \text{ Quadrant}$$
$$z^3 = 2\sqrt{2}\left(\cos -\frac{\pi}{4} + i\sin -\frac{\pi}{4}\right)$$
$$\mathbf{z^3 = 2 - 2i}$$

$$z^5 = (\sqrt{2})^5 e^{i\left(-\frac{3\pi}{4}\right)\cdot 5}$$
$$z^5 = 4\sqrt{2}e^{i\left(-\frac{15\pi}{4}\right)}$$
$$z^5 = 4\sqrt{2}e^{i\left(\frac{\pi}{4}\right)} \rightarrow 1^{\text{st}} \text{ Quadrant}$$
$$z^5 = 4\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$
$$\mathbf{z^5 = 4 + 4i}$$

$$z^7 = (\sqrt{2})^7 e^{i\left(-\frac{3\pi}{4}\right)\cdot 7}$$
$$z^7 = 8\sqrt{2}e^{i\left(-\frac{21\pi}{4}\right)}$$
$$z^7 = 8\sqrt{2}e^{i\left(\frac{3\pi}{4}\right)} \rightarrow 2^{\text{nd}} \text{ Quadrant}$$
$$z^7 = 8\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$
$$\mathbf{z^7 = -8 + 8i}$$

3. Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ x & 0 & 1 & x \\ 1 & 0 & x & 0 \\ x & 0 & 0 & 1 \end{bmatrix}, \text{ where x is a real number.}$$

a. Show that determinant of A is $x^3 - x^2 + 1$.

The determinant of matrix A can be found by summing the products of terms of any row/column with the corresponding cofactors. For this question, take the second column:

$$\det(A) = a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32} + a_{42}C_{42}$$

$$*[a_{22} = a_{42} = a_{42} = 0]$$

$$\det(A) = 1.C_{12} + 0.C_{22} + 0.C_{32} + 0.C_{42}$$

$$\det(A) = C_{12}$$

$$\det(A) = (-1)^{1+2} . \begin{vmatrix} x & 1 & x \\ 1 & x & 0 \\ x & 0 & 1 \end{vmatrix}$$

$$*[\text{det of 3x3 matrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}]$$

$$\det(A) = (-1)^3 . (x.x.1 + 1.0.x + x.1.0 - x.0.0 - 1.1.1 - x.x.x)$$

$$\det(A) = (-1) . (x^2 + 0 + 0 - 0 - 1 - x^3)$$

$$\mathbf{\det(A) = x^3 - x^2 + 1}$$

<*shown*>

b. Show that A is a singular for some value of $x \in (-1,0)$. State the theorem used.

Matrix A is singular if and only if $\det(A) = 0$ for some value of $x \in (-1,0)$.
Therefore, we need to show that $x^3 - x^2 + 1 = 0$ for some value of $x \in (-1,0)$.

[Proof] Let $f(x) = x^3 - x^2 + 1 = 0$ for some value of $x \in (-1,0)$. The function f is continuous on $[-1,0]$, since it is a polynomial function. Note that,

$$f(-1) = -1 < 0 \text{ and } f(0) = 1 > 0.$$

By Intermediate Value Theorem, there is a real number $c \in (-1,0)$ such that $f(c) = 0$. Thus, $\det(A) = 0$.

∴Therefore, **A is a singular for some value of $x \in (-1,0)$.** <*shown*>

4. a. Figure 3 shows the graph of a differentiable function $f(x)$. Given that (1,–2) and (–1,2) are the local minimum and maximum of the graph, and the graph has a minimum gradient of –3 at $x = 0$, **sketch the graph of $f'(x)$ on the same diagram** (Figure 3).

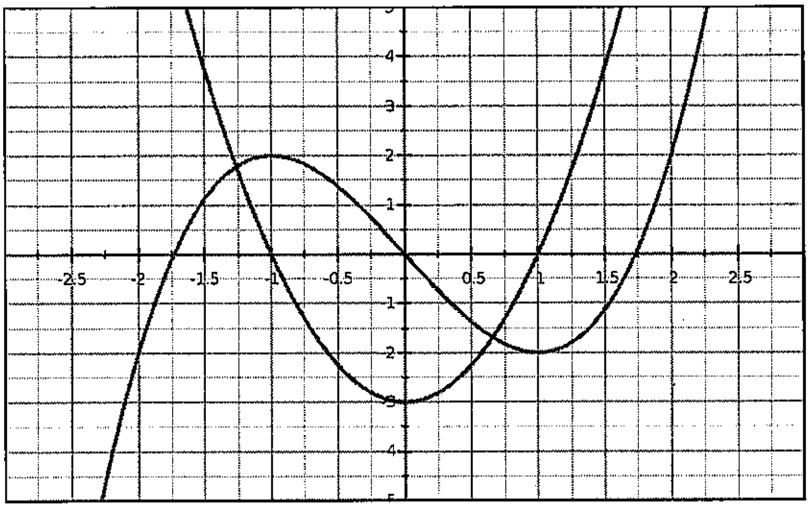


Figure 3

The equation $f(x)$ is a cubic polynomial, while $f'(x)$ is a quadratic polynomial.
Let $f'(x) = ax^2 + bx + c$,

- $f'(0) = -3 \rightarrow c = -3$
- $f'(1) = a + b - 3 = 0 \rightarrow a + b = -3$ … (1)
- $f'(-1) = a - b - 3 = 0 \rightarrow a - b = -3$ … (2)

By solving (1) and (2), we get $a = 3$ and $b = 0$. Therefore **$f'(x) = 3x^2 - 3$.**
Critical points to sketch the $f'(x)$ graph: **$f'(0) = -3$, $f'(1) = 0$, $f'(-1) = 0$.**

b. Figure 4 is the graph of a piecewise linear function $f(x)$ (i.e., the graph of $y = f(x)$ is made up of straight lines).

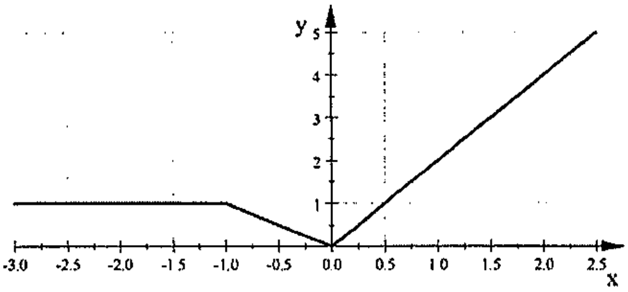


Figure 4

i. State the definition of the derivative $f'(c)$ of $f(x)$ at $x = c$. Use the **definition of derivative** to show that $f'(0)$ does not exist.

The derivative $f'(c)$ of $f(x)$ at $x = c$ is defined as:

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

$$f(x) = \begin{cases} -x, & x < 0 \\ 2x, & x \geq 0 \end{cases}$$

To show that $f'(0)$ does not exist using the definition of derivative, we need to show that the limit $\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ does not exist. To prove whether the limit exist or not, we use the equal one-sided limits theorem:

$$\lim_{x \to a} f(x) = L \text{ if and only if } \lim_{x \to a^-} f(x) = L \text{ and } \lim_{x \to a^+} f(x) = L$$

Note that $f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{f(x)}{x}$.

Thus, we have to consider both $\lim_{x \to 0^-} \frac{f(x)}{x}$ and $\lim_{x \to 0^+} \frac{f(x)}{x}$, and check whether these limits are equal.

- $\lim_{x \to 0^+} \frac{f(x)}{x} = \lim_{x \to 0^+} \frac{2x}{x} = 2$
- $\lim_{x \to 0^-} \frac{f(x)}{x} = \lim_{x \to 0^-} \frac{-x}{x} = -1$

Since $\lim_{x \to 0^-} \frac{f(x)}{x} \neq \lim_{x \to 0^+} \frac{f(x)}{x}$, we conclude that $\lim_{x \to 0} \frac{f(x)}{x}$ does not exist.

∴Therefore, $f'(0)$ does not exist. <*shown*>

ii. Evaluate $\int_{-2}^{2} f(x)\, dx$.

To evaluate the integral above is the same as finding the area under the curve.

- I : $-2 < x < -1$ → 1 x 1 = 1 units^2
- II : $-1 < x < 0$ → (1 x 1) / 2 = 0.5 units^2
- III : $0 < x < 1$ → (2 x 4) / 2 = 4 units^2

The total area under the curve is: I + II + III = 1 + 0.5 + 4 = 5.5 units^2

∴Therefore, $\int_{-2}^{2} f(x)\, dx$ **= 5.5 units^2**

5. a.

i. State, without proof, the Mean Value Theorem.

Mean Value Theorem:

Let f be continuous on the closed interval [a,b] and differentiable on the open interval (a,b). Then there is (at least one point) c in (a,b) such that

$$\frac{f(b)-f(a)}{b-a} = f'(c).$$

ii. If $f'(x) > 0$ for all $x \in \mathbb{R}$, **use the Mean Value Theorem** to show that f is an increasing function.

Suppose $f'(x) > 0$ for all $x \in \mathbb{R}$. Let x_1, x_2 be in $\mathbb{R}$ with $x_1 < x_2$. So, the closed interval $[x_1, x_2]$ is contained in $\mathbb{R}$, and the open interval (x_1, x_2) is contained in $\mathbb{R}$. So, f is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2). The Mean Value Theorem applies. By Mean Value Theorem, then there is c between x_1 and x_2, such that

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(c) > 0$$

Since $x_2 - x_1 > 0$, we multiply the equation by $x_2 - x_1$, and we now have $f(x_2) - f(x_1) > 0$ and $f(x_2) > f(x_1)$. Since x_1 and x_2 were arbitrary members of $\mathbb{R}$ with $x_1 < x_2$, this means f **is increasing on** $\mathbb{R}$. <*shown*>

b. Compute the following limits.

i. $\lim_{x\to-\infty}\frac{x}{\sqrt{x^2+1}}$

Note that for $x < 0$ we have $x = -\sqrt{x^2}$.

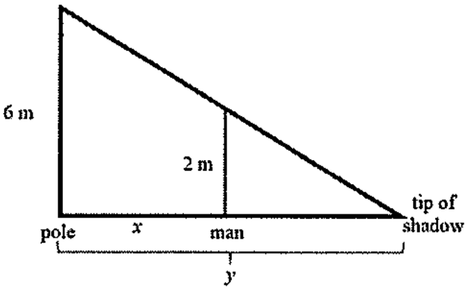
$$\lim_{x\to-\infty}\frac{x}{\sqrt{x^2+1}} = \lim_{x\to-\infty}\frac{-\sqrt{x^2}}{\sqrt{x^2+1}} = \lim_{x\to-\infty} -\sqrt{\frac{x^2}{x^2+1}} = \lim_{x\to-\infty} -\sqrt{\frac{1}{1+\frac{1}{x^2}}} = \mathbf{-1}$$

ii. $\lim_{x\to1}\left(\frac{1}{\ln x}-\frac{x}{x-1}\right)$

$$\lim_{x\to1}\left(\frac{1}{\ln x}-\frac{x}{x-1}\right) = \lim_{x\to1}\overbrace{\left(\frac{x-1-x.\ln x}{(x-1).\ln x}\right)}^{\frac{0}{0}} \underset{L'Hrule}{=} \lim_{x\to1}\left(\frac{1-\ln x-1}{\ln x+\frac{x-1}{x}}\right) = \lim_{x\to1}\underbrace{\left(\frac{-\ln x}{\ln x+1-\frac{1}{x}}\right)}_{\frac{0}{0}}$$

$$\underset{L'Hrule}{=} \lim_{x\to1}\left(\frac{-\frac{1}{x}}{\frac{1}{x}+\frac{1}{x^2}}\right) = -\frac{\mathbf{1}}{\mathbf{2}}$$

6. a. A street light is mounted at the top of a 6-metre-tall pole. A man 2 m tall is walking away from the pole at a speed of 2 m/s along a straight path. How fast is the tip of his shadow moving when he is 10 m away from the pole?



The speed of the tip of his shadow when he is 10 m away from the pole: $\frac{dy}{dt}_{y=10} = ?$

According to triangle proportionality theorem, $\frac{y-x}{y} = \frac{2\,m}{6\,m}$

$$\frac{y-x}{y} = \frac{1}{3} \Leftrightarrow y = 3y - 3x \Leftrightarrow 2y = 3x \Leftrightarrow y = \frac{3}{2}x$$

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{3}{2}$$

Using chain rule, (speed of man $\frac{dx}{dt} = 2$)

$$\frac{dy}{dt} = \frac{dy}{dx}.\frac{dx}{dt} \Leftrightarrow \frac{dy}{dt} = \frac{3}{2}.2 = 3\,m/s$$

$\therefore$ *The tip of his shadow is moving with the speed of* $\mathbf{3}\ \boldsymbol{m/s}$ *when he is* $10\,m$ *away from the pole.*

b. Find the global maximum and minimum of the function $f(x) = ln(x^2 + 1) + x$ on the interval [–2,2].

$$f'(x) = \frac{2x}{x^2+1} + 1 = \frac{2x}{x^2+1} + \frac{x^2+1}{x^2+1} = \frac{x^2+2x+1}{x^2+1} = \frac{(x+1)^2}{x^2+1}$$

Note that there is no singular point of f in [–2,2].

Setting $f'(x) = 0$, we have x = – 1, which gives a stationary point of f.

Thus, x = – 1 is the only critical point of f in [–2,2].

For both $-2 < x < -1$ and $-1 < x < 2$, $f'(x) > 0$. By the first derivative test, $f(-1)$ is an inflection point.

➔ $f(-2) = \ln 5 - 2 \approx -0.39$; $f(2) = \ln 5 + 2 \approx 3.61$; $f(-1) = \ln 2 - 1 \approx -0.31$

$\therefore$Therefore, we conclude that **the global maximum of the function f(x) on the interval [–2,2] is f(2) and the global minimum of the function f(x) on the interval [–2,2] is f(–2).**

7. a. Evaluate the following integrals.

i. $\int \frac{x+2}{x^2+5x-6}\,dx$

$$\int \frac{x+2}{x^2+5x-6}\,dx$$

$$Write\ x+2\ as\ \frac{1}{2}(2x+5)\ +\ (-\frac{1}{2})\ and\ split:$$

$$=\frac{1}{2}\int \frac{2x+5}{x^2+5x-6}\,dx-\frac{1}{2}\int \frac{1}{x^2+5x-6}\,dx$$

$$Solving:$$

$$\frac{1}{2}\int \frac{1}{x^2+5x-6}\,dx$$

$$Factor\ the\ denominator:$$

$$=\frac{1}{2}\int \frac{1}{(x-1)(x+6)}\,dx$$

$$Perform\ partial\ fraction\ decomposition:$$

$$=\frac{1}{2}\int \frac{1}{7(x-1)}-\frac{1}{7(x+6)}\,dx$$

$$\int \frac{x+2}{x^2+5x-6}\,dx$$

$$=\frac{1}{2}\int \frac{2x+5}{x^2+5x-6}\,dx-\frac{1}{2}\int \frac{1}{x^2+5x-6}\,dx+C$$

$$=\frac{1}{2}\int \frac{2x+5}{x^2+5x-6}-\left(\frac{1}{7(x-1)}-\frac{1}{7(x+6)}\right)dx+C$$

$$=\frac{ln|x^2+5x-6|}{2}+\frac{ln|x+6|}{14}-\frac{ln|x-1|}{14}+C$$

$$=\frac{\mathbf{4\,ln|x+6|}}{\mathbf{7}}-\frac{\mathbf{3\,ln|x-1|}}{\mathbf{7}}+\boldsymbol{C}$$

ii. $\int x\, tan^{-1} x \ dx$

We use integration by parts: $u = \tan^{-1} x \ and\ v' = x\ so\ that\ u' = \frac{1}{1+x^2}\ and\ v = \frac{x^2}{2}$.

$$\int x\, tan^{-1} x \ dx$$

$$= \frac{x^2}{2}\tan^{-1} x - \int \frac{x^2}{2}\frac{1}{1 + x^2}dx$$

$$= \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2}\int 1 - \frac{1}{1 + x^2}dx$$

$$= \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2}(\mathrm{x} - \tan^{-1} x\,) + C$$

$$= \boldsymbol{\frac{1}{2}\left((x^2 + 1)\tan^{-1} x - x\right) + C}$$

b. Find the values of p for which the improper integral $\int_e^{\infty} \frac{1}{x(ln\,x)^p} dx$ converges and evaluate the integrals for those values of p.

$$\int_e^{\infty} \frac{1}{x(ln\,x)^p} dx = \lim_{t\to\infty}\int_e^{t} \frac{1}{x(ln\,x)^p} dx$$

$$= \lim_{t\to\infty}\begin{cases} \frac{(ln\,t)^{1-p}}{1-p} - \frac{(ln\,e)^{1-p}}{1-p}, & if\ p < 1 \\ \frac{(ln\,t)^{1-p}}{1-p} - \frac{(ln\,e)^{1-p}}{1-p}, & if\ p > 1 \\ ln|ln\,t| - ln|ln\,e|, & if\ p = 1 \end{cases}$$

$$= \lim_{t\to\infty}\begin{cases} diverge, & if\ p < 1 \\ -\frac{(ln\,e)^{1-p}}{1-p}, & if\ p > 1 \\ diverge, & if\ p = 1 \end{cases}$$

∴The improper integral $\int_e^{\infty} \frac{1}{x(ln\,x)^p} dx$ converges when p > 1.

$$\int_e^{\infty} \frac{1}{x(ln\,x)^p} dx = \lim_{t\to\infty}\int_e^{t} \frac{1}{x(ln\,x)^p} dx = \lim_{t\to\infty} -\frac{(ln\,e)^{1-p}}{1-p} = -\boldsymbol{\frac{1}{1-p}}$$

$$\therefore \int_e^{\infty} \boldsymbol{\frac{1}{x(ln\,x)^p} dx = -\frac{1}{1-p}}$$

c. Use Simpson's Rule and Trapezoidal Rule with n = 10 to approximate $\int_0^1 \frac{1}{x+1}dx$. Show that S_{10} is a better approximation to the actual value of $\int_0^1 \frac{1}{x+1}dx$ than T_{10}.

$$\Delta x = h = \frac{(1-0)}{10} = 0.1$$

$$f(x) = \frac{1}{x+1}$$

$$T_{10} = \int_0^1 \frac{1}{x+1}dx$$

$$\approx \frac{\Delta x}{2}(y_0 + 2y_1 + 2y_2 + 2y_3 + 2y_4 + 2y_5 + 2y_6 + 2y_7 + 2y_8 + 2y_9 + y_{10})$$

$$= \frac{0.1}{2}\left(1 + 2\cdot\frac{10}{11} + 2\cdot\frac{5}{6} + 2\cdot\frac{10}{13} + 2\cdot\frac{5}{7} + 2\cdot\frac{2}{3} + 2\cdot\frac{5}{8} + 2\cdot\frac{10}{17} + 2\cdot\frac{5}{9} + 2\cdot\frac{10}{19} + \frac{1}{2}\right)$$

$$= 0.6937714032$$

$$S_{10} = \int_0^1 \frac{1}{x+1}dx$$

$$\approx \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + 2y_6 + 4y_7 + 2y_8 + 4y_9 + y_{10})$$

$$= \frac{0.1}{3}\left(1 + 4\cdot\frac{10}{11} + 2\cdot\frac{5}{6} + 4\cdot\frac{10}{13} + 2\cdot\frac{5}{7} + 4\cdot\frac{2}{3} + 2\cdot\frac{5}{8} + 4\cdot\frac{10}{17} + 2\cdot\frac{5}{9} + 4\cdot\frac{10}{19} + \frac{1}{2}\right)$$

$$= 0.6931502307$$

k	x_k	$y_k = f(x_k)$
0	0	1
1	0.1	10/11
2	0.2	5/6
3	0.3	10/13
4	0.4	5/7
5	0.5	2/3
6	0.6	5/8
7	0.7	10/17
8	0.8	5/9
9	0.9	10/19
10	1	1/2

The exact value of the definite integral $\int_0^1 \frac{1}{x+1}dx$ is ln 2 (0.6931471806).
The relative error is thus,

$$\left|\frac{T_{10} - I}{I}\right| \approx 0.0009005629 \text{ or } 0.09005629\%$$

$$\left|\frac{S_{10} - I}{I}\right| \approx 0.0000044004 \text{ or } 0.00044004\%$$

∴ **Therefore, S_{10} is a better approximation to the actual value of $\int_0^1 \frac{1}{x+1}dx$ than T_{10}.** ***<shown>***

NANYANG TECHNOLOGICAL UNIVERSITY
SEMESTER 2 EXAMINATION 2017-2018
MH1810 - Mathematics 1

May 2018 TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **EIGHT (8)** questions and comprises **SEVEN (7)** printed pages.
2. Answer **ALL** questions. The marks for each question are indicated at the end of each question.
3. Answer each question beginning on a **FRESH** page of the answer book.
4. This **IS NOT** an **OPEN BOOK** exam.
5. Candidates may use calculators. However, they should write down systematically the steps in the workings.

QUESTION 1. Let $z = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$.

(a) Find the modulus and the principal argument of z. **(7 marks)**

(b) Using part (a) and De Moivre's Theorem, find z^5. Express your answer in the form $x + yi$, where x and y are real numbers. **(8 marks)**

QUESTION 2. Let $\mathbf{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

(a) Compute $\mathbf{u} \cdot \mathbf{v}$ **(5 marks)**

(b) Compute $\mathbf{u} \times \mathbf{v}$ **(5 marks)**

QUESTION 3. Let $B = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 19 & 6 & 0 \end{pmatrix}$.

(a) Calculate the determinant of B via cofactor expansion along the second row. **(5 marks)**

(b) Decide whether B is invertible or not. Justify your answer. **(5 marks)**

QUESTION 4. Find the following limits

(a) $\lim\limits_{x\to 1} \dfrac{x^2-1}{x-1}$ **(8 marks)**

(b) $\lim\limits_{x\to\infty} \dfrac{3x^2+1}{1-x^2}$ **(7 marks)**

QUESTION 5. Use the squeeze theorem to find the following limit:

$$\lim_{x\to 0}\left(e^{x^2}-1\right)\sin(x).$$

(10 marks)

QUESTION 6. Suppose that a function f is continuous on the closed interval $[0,3]$ and $0 \le f(x) \le 3$ for every $x \in [0,3]$. Is it true that $f(c) = c$ for some $c \in [0,3]$? Justify your answer. **(10 marks)**

QUESTION 7. Let $f(x) = (x^2-1)^{2/3}$. Find the global maximum and global minimum values of f on the interval $[-3,3]$. **(10 marks)**

QUESTION 8.

(a) Evaluate the integral $\displaystyle\int \frac{x+4}{x^3+3x^2-10x}\,dx$. **(6 marks)**

(b) Evaluate the integral $\displaystyle\int \frac{1}{\sqrt{4+x^2}}\,dx$. **(7 marks)**

(c) Determine whether $\displaystyle\int_2^\infty \frac{1}{\ln x}\,dx$ converges or diverges. Justify your answer. **(7 marks)**

Appendix

Numerical Methods.

- Linearization Formula:

$$L(x) = f(a) + f'(a)\,(x-a)$$

- Newton's Method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

- Trapezoidal Rule:

$$\int_a^b f(x)\,dx \approx T_n = \frac{h}{2}\left[y_0 + 2\,(y_1 + y_2 + \cdots + y_{n-1}) + y_n\right]$$

- Simpson's Rule:

$$\int_a^b f(x)\,dx \quad\approx\quad S_n = \frac{h}{3}\left[y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n\right],$$

where n is even.

Derivatives.

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\sinh x) = \cosh x$$

$$\frac{d}{dx}(\tanh x) = \mathrm{sech}^2 x$$

$$\frac{d}{dx}(\mathrm{sech}\ x) = -\mathrm{sech} x \tanh x$$

$$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{x^2+1}}$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(a^x) = a^x \ln a$$

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\cosh x) = \sinh x$$

$$\frac{d}{dx}(\coth x) = -\mathrm{csch}^2 x$$

$$\frac{d}{dx}(\mathrm{csch}\ x) = -\mathrm{csch} x \coth x$$

Antiderivatives.

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,\, n \neq -1 \qquad \int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \csc^2 x\,dx = -\cot x + C$$

$$\int \tan x \sec x\,dx = \sec x + C \qquad \int \cot x \csc x\,dx = -\csc x + C$$

$$\int \tan x\,dx = \ln|\sec x| + C \qquad \int \cot x\,dx = \ln|\sin x| + C$$

$$\int e^x\,dx = e^x + C \qquad \int a^x\,dx = \frac{a^x}{\ln a} + C,\, a > 0$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C \qquad \int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C,\, |x| < |a| \qquad \int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{\sqrt{x^2+1}}\,dx = \sinh^{-1} x + C \qquad \int \frac{1}{\sqrt{x^2+a^2}}\,dx = \sinh^{-1}\left(\frac{x}{a}\right) + C$$

END OF PAPER

MH1810 MATHEMATICS 1

Please read the following instructions carefully:

1. **Please do not turn over the question paper until you are told to do so. Disciplinary action may be taken against you if you do so.**

2. You are not allowed to leave the examination hall unless accompanied by an invigilator. You may raise your hand if you need to communicate with the invigilator.

3. Please write your Matriculation Number on the front of the answer book.

4. Please indicate clearly in the answer book (at the appropriate place) if you are continuing the answer to a question elsewhere in the book.

NANYANG TECHNOLOGICAL UNIVERSITY
SEMESTER 2 EXAMINATION 2017-2018
MH1810 - Mathematics 1

May 2018

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(a) $\lim_{x\to 1} \frac{x^2-1}{x-1}$ (8 marks)

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(10 marks)

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QUESTION 7. Let $f(x) = (x^2-1)^{2/3}$. Find the global maximum and global minimum values of f on the interval $[-3,3]$. (10 marks)

QUESTION 8.

(a) Evaluate the integral $\int \frac{x+4}{x^3+3x^2-10x}\,dx$. (6 marks)

(b) Evaluate the integral $\int \frac{1}{\sqrt{4+x^2}}\,dx$. (7 marks)

(c) Determine whether $\int_2^\infty \frac{1}{\ln x}\,dx$ converges or diverges. Justify your answer. (7 marks)

Subject Code : MH1810

Subject Name : Mathematics 1

Year / Semester : 2017-2018 SEM 2

Suggested Solution Brought to You by Materials Science and Engineering Club

1a) $z = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$

$|z| = \sqrt{\left(-\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2}$

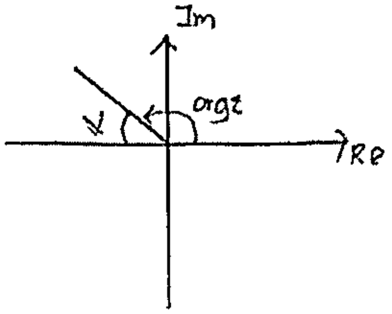
$= 1$

Basic angle, $\alpha = \tan^{-1}\left(\frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}\right)$

$= \tan^{-1} 1$

$= \frac{\pi}{4}$

$\arg(z) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$ *



b) $z^5 = |z|^5$

$= 1^5$

$= 1$

$\arg(z^5) = 5 \cdot \frac{3\pi}{4}$

$= \frac{15\pi}{4}$

$z^5 = \cos\frac{15\pi}{4} + i\sin\frac{15\pi}{4}$

$= \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$

De Moivre's Theorem

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

✳ Could Use GrC to check the answers

$\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)^5 = 0.7071106 - 0.7071106i$

↑

Type/Insert

Subject Code : MH1810

Subject Name : Mathematics 1

Year / Semester : 2017-2018 SEM 2

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2a) $u = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad v = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$$u \cdot v = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$= 1\cdot1 + 1\cdot1 + 1\cdot0$$

$$= 2$$

b)

$$u \times v = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} i - \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} j + \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} k$$

$$= (1\cdot0 - 1\cdot1) i - (1\cdot0 - 1\cdot1) j + (1\cdot1 - 1\cdot1) k$$

$$= -i - j$$

$$= \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$$

Subject Code : MH1810

Subject Name : Mathematics I

Year / Semester : 2017-2018 SEM 2

Suggested Solution Brought to You by Materials Science and Engineering Club

3a) $B=\begin{pmatrix}1&0&1\\1&1&2\\19&6&0\end{pmatrix}$

$$\det B=-1\cdot\begin{vmatrix}0&1\\6&0\end{vmatrix}+1\cdot\begin{vmatrix}1&1\\19&0\end{vmatrix}-2\begin{vmatrix}1&0\\19&6\end{vmatrix}$$

$$=-1(0-6)+1(0-19)-2(6-0)$$

$$=-25$$

Double check your answer by

① With another "easy" row or column [OR]

② Using GC,

press $\boxed{2nd}$ $\boxed{x^{-1}}$ and edit matrix

press $\boxed{2nd}$ $\boxed{mode}$ after filling the matrix

Go to the matrix, under math,
use "det(", and your matrix,
you will get the same answer.

b)

Since $\det B\neq 0$, B is invertible

①Find cofactor

②Find adj(B) by transpose C

$$\begin{pmatrix}+&-&+\\-&+&-\\+&-&+\end{pmatrix}$$

$$C=\begin{bmatrix}-12&+38&-13\\+6&-19&-6\\-1&-1&1\end{bmatrix}$$

$$\operatorname{adj}(B)=C^T$$

$$=\begin{bmatrix}-12&6&-1\\38&-19&-1\\-13&-6&1\end{bmatrix}$$

$$B\cdot\operatorname{adj}(B)=\begin{pmatrix}1&0&1\\1&1&2\\19&6&0\end{pmatrix}\begin{pmatrix}-12&6&-1\\38&-19&-1\\-13&-6&1\end{pmatrix}$$

$$=\begin{pmatrix}-25&0&0\\0&-25&0\\0&0&-25\end{pmatrix}=-25I$$

※

Subject Code : MH1810

Subject Name : Mathematics I

Year / Semester : 2017-2018 SEM 2

Suggested Solution Brought to You by Materials Science and Engineering Club

4a)

$$\lim_{x\to1}\frac{x^2-1}{x-1}=\lim_{x\to1}\frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}}$$

$$=\lim_{x\to1} x+1$$

$$=2 \divideontimes$$

$x^2-1=(x+1)(x-1)$

b)

$$\lim_{x\to\infty}\frac{3x^2+1}{1-x^2}=\lim_{x\to\infty}\left(-3+\frac{4}{1-x^2}\right)$$

$$=-3$$

$$\begin{array}{r|l} & \quad\quad\quad -3 \\ \hline -x^2+1 & 3x^2+0x+1 \\ & 3x^2 \quad\quad -3 \\ \hline & \quad\quad\quad 4 \divideontimes \end{array}$$

5)

$$-1\le\sin(x)\le1$$

$$-(e^{x^2}-1)\le\left(e^{x^2}-1\right)\cdot\sin(x)\le e^{x^2}-1$$

For any values of x,

$e^{x^2}-1\geq0$

↳ always +ve

upper limit $\lim_{x\to0}\left(e^{x^2}-1\right)=e^0-1$
$=0$

lower limit $\lim_{x\to0}-(e^{x^2}-1)=-(e^0-1)$
$=0$

By squeeze theorem, $\lim_{x\to0}\left(e^{x^2}-1\right)\cdot\sin x=0$ $\divideontimes$

Subject Code : MH1810	Suggested Solution Brought to You by Materials Science and Engineering Club
Subject Name : Mathematics I	
Year / Semester : 2017-2018 SEM 2	

6) $D_f = [0,3]$

$R_f = [0,3]$

Since the function f is continuous,

$$\lim_{x \to c} f(x) = f(c)$$

With D_f and R_f having the same range of values,

It is possible $f(c) = c$ for some $c \in [0,3]$

* Not very sure how to answer such question, can consult TA, Prof to find out the correct way to present it.

Subject Code	: MH1810	**Suggested Solution Brought to You by Materials Science and Engineering Club**
Subject Name	: Mathematics I	
Year / Semester	: 2017-2018 SEM 2	

7) $f(x) = (x^2-1)^{\frac{2}{3}}$

$f'(x) = \frac{2}{3}(x^2-1)^{-\frac{1}{3}} \cdot (2x)$

$= \frac{4x}{3\sqrt{x^2-1}} \cdot \frac{1}{3}$

$f'(x)=0$ for stationary point,

$x=0$

$f(0) = (-1)^{\frac{2}{3}}$

$= 1$

When $y=0$,

$(x^2-1)^{\frac{2}{3}} = 0$

$[(x+1)(x-1)]^{\frac{2}{3}} = 0$

$x = \pm 1$

When $x = \pm 3$,

$y = 4$

Points $(0,1)$; $(-1,0)$; $(1,0)$; $(-3,4)$; $(3,4)$

Global minimum of f $(-1,0)$; $(1,0)$

Global maximum of f $(-3,4)$; $(3,4)$

Using GC graph function will benefit a lot for this questions

Subject Code : MH1810	**Suggested Solution Brought to You by Materials Science and Engineering Club**
Subject Name : Mathematics I	
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8) $x^3+3x^2-10x = x(x^2+3x-10)$

$= x(x+5)(x-2)$

$$\begin{array}{cc|c} x & 5 & 5x \\ x & -2 & -2x \\ \hline x^2 & -10 & 3x \checkmark \end{array}$$

$$\frac{x+4}{x^3+3x^2-10x} = \frac{A}{x} + \frac{B}{x+5} + \frac{C}{x-2} = \frac{A(x+5)(x-2)+B(x)(x-2)+C(x)(x+5)}{x^3+3x+10x}$$

sub $x=0$ $\quad 4 = A(5)(-2) \quad A = -\frac{2}{5}$

sub $x=2$ $\quad 2+4 = C(2)(7) \quad C = \frac{3}{7}$

sub $x=-5$ $\quad -5+4 = B(-5)(-5-2) \quad B = -\frac{1}{35}$

$$\int \frac{x-4}{x^3+3x^2-10x}\,dx = \int -\frac{2}{5x} - \frac{1}{35(x+5)} + \frac{3}{7(x-2)}\,dx$$

$$= -\frac{2}{5}\ln x - \frac{1}{35}\ln(x+5) + \frac{3}{7}\ln(x-2) + C$$

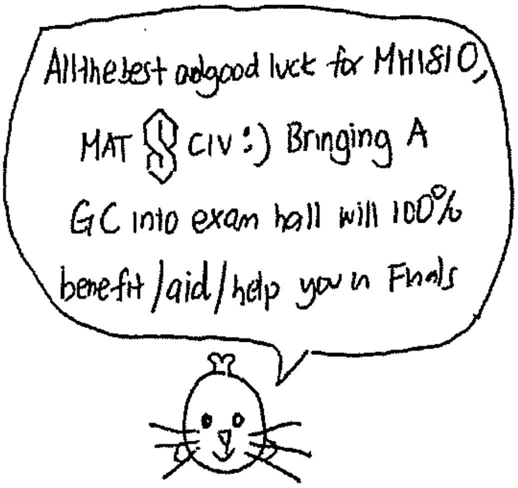
b) $\int \frac{1}{\sqrt{4-x^2}}\,dx = \sinh^{-1}\left(\frac{x}{2}\right) + C$ (from formula sheet)

c) For $x>2$, $\ln x < x \rightarrow \frac{1}{\ln x} > \frac{1}{x}$

$$\int_2^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} (\ln t - \ln 2) = \infty$$

By comparison theorem,

$$\int_2^\infty \frac{1}{\ln x}\,dx \text{ diverges.}$$



NANYANG TECHNOLOGICAL UNIVERSITY
SEMESTER I EXAMINATION 2019–2020
MH1810 – Mathematics 1

NOVEMBER 2019 TIME ALLOWED: 2 HOURS

Matriculation Number:

Seat Number:

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **EIGHT (8)** questions and comprises **TWENTY (20)** pages, including an Appendix.
2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
3. This **IS NOT** an **OPEN BOOK** exam. However, a list of formulae is provided in the Appendix.
4. Candidates may use calculators. However, they should write down systematically the steps in the workings.
5. All your solutions should be written in this booklet within the space provided after each question. If you use an additional answer book, attach it to this booklet and hand them in at the end of the examination.

For examiners only

Questions	Marks
1 (10)	
2 (10)	
3 (10)	
4 (15)	

Questions	Marks
5 (15)	
6 (15)	
7 (15)	
8 (10)	

Total (100)	

QUESTION 1. **(10 Marks)**

Three points A, B, C and the vector $\overrightarrow{AB}$ are shown in Figure 1. Given that $\overrightarrow{OA} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \overrightarrow{OB} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \overrightarrow{OC} = \begin{pmatrix} -9 \\ -6 \\ -3 \end{pmatrix}$ and $\overrightarrow{OD} = \begin{pmatrix} -3 \\ 0 \\ 3 \end{pmatrix}$.

(a) Draw the vector $\overrightarrow{CD}$ in Figure 1.

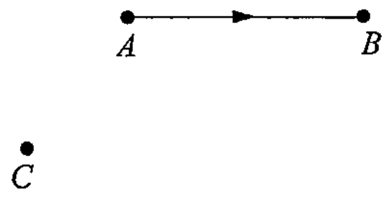


Figure 1.

Question 1 continues on Page 3.

(b) Find the area of quadrilateral $ABDC$. Express the answer in the form $m\sqrt{n}$, where m and n are integers to be determined.

QUESTION 2. **(10 Marks)**

(a) Consider the complex number

$$w = \frac{\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)^2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^3}{\left(\cos\frac{\pi}{12} - i\sin\frac{\pi}{12}\right)^2}.$$

(i) Find the modulus and principal argument of w.

(ii) Use De Moivre's Theorem to show that w is a cube root of 1.

Question 2 continues on Page 5.

(b) Find the value of the complex number z such that

$$z + |z+12| = 6 + 6i.$$

Indicate on Figure 2 the complex numbers $-z$, iz, and z^*.

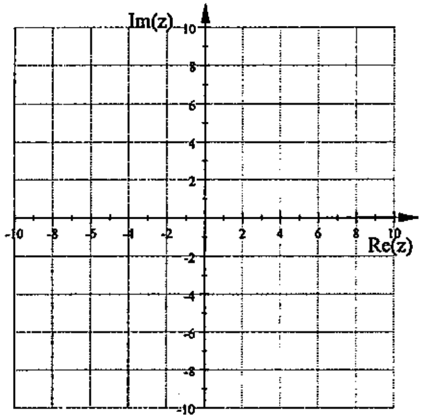


Figure 2

QUESTION 3. **(10 Marks)**

The graph of a cubic polynomial $f(x) = ax^3 + bx^2 + cx$ has turning points $(1, -7)$ and $(-2, 20)$.

(a) Show that a, b and c satisfy the following two equations.

$$12a - 4b + c = 0 \text{ and } 3a + 2b + c = 0.$$

Find another two equations relating a, b and c.

Question 3 continues on Page 7.

(b) By using *cofactor expansion along the third row*, show that the determinant of the following matrix is 21.

$$\begin{pmatrix} 12 & -4 & 1 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

(c) By using Cramer's Rule, find the value of c.

QUESTION 4. **(15 Marks)**

(a) Let $f(x) = x^{2/3}$, using *the definition of derivative*, find $f'(x)$.

(b) Let

$$f(x) = \begin{cases} x^3 + x + 1 & \text{if } x \geq 0 \\ a \sin x + b & \text{if } x < 0 \end{cases},$$

where a and b are some constants. Find the values of a and b so that f is differentiable.

Question 4 continues on Page 9.

(c) A curve is defined implicitly by the equation

$$x^{2/3} + y^{2/3} = 4.$$

The point $P\,(x_0, y_0)$ lies on the curve.

(i) Find $\dfrac{dy}{dx}$ in terms of x and y.

(ii) Show that the tangent of the curve at the point $P\,(x_0, y_0)$ is

$$\frac{x}{x_0^{1/3}} + \frac{y}{y_0^{1/3}} = 4.$$

QUESTION 5 **(15 Marks)**

Find the following limits.

(a) $\displaystyle\lim_{x\to 1^+} \frac{(x^2-1)\,e^x}{\sqrt{x}-1}$

(b) $\displaystyle\lim_{x\to 0} \frac{\sin x}{\sqrt{x+20}+\sin\frac{1}{x}}$

Question 5 continues on Page 11.

(c) $\lim_{x \to 0^+} x^{x^2}$

QUESTION 6. **(15 Marks)**

(a) The volume of a metal cube is increasing at a rate of 10 cm^3 / min. How fast is the surface area increasing when the length is 30 cm?

Question 6 continues on Page 13.

(b) A boat left a jetty at 1:00 PM and travelled due north at 40 km/h. Another boat was travelling due west at 30 km/h and reached the same jetty at 2:00 PM. At what time were the two boats nearest to each other? Justify your answer.

QUESTION 7. **(15 Marks)**

Evaluate each of the following.

(a) $\int \frac{x+2}{\sqrt{7-2x^2+4x}}\,dx.$

(b) $\int \frac{1}{e^x+1}dx.$

Question 7 continues on Page 15.

(c) $\lim\limits_{n\to\infty} \dfrac{1}{\sqrt{n}} \left(\dfrac{1}{\sqrt{n+1}} + \dfrac{1}{\sqrt{n+2}} + \cdots + \dfrac{1}{\sqrt{n+n}} \right)$

QUESTION 8. **(10 Marks)**

The region R is enclosed by the lines $y = 0$, $x = 1$, $x = 2$ and the graph of $y = \frac{1}{4+x^2}$.

(a) Find the area of R.

Question 8 continues on Page 17.

(b) Find the volume of the solid obtained when the region R is rotated about the line $x = 1$ by 2π radians.

END OF PAPER

Appendix

Numerical Methods.

- Linearization Formula:

$$L(x) = f(a) + f'(a)\,(x - a)$$

- Newton's Method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

- Trapezoidal Rule:

$$\int_a^b f(x)\,dx \approx T_n = \frac{h}{2}\left[y_0 + 2\,(y_1 + y_2 + \cdots + y_{n-1}) + y_n\right]$$

- Simpson's Rule:

$$\int_a^b f(x)\,dx \;\approx\; S_n = \frac{h}{3}\left[y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n\right],$$

where n is even.

Derivatives.

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(e^x) = e^x \qquad \frac{d}{dx}(a^x) = a^x \ln a$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x} \qquad \frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\cosh x) = \sinh x$$

$$\frac{d}{dx}(\tanh x) = \text{sech}^2 x \qquad \frac{d}{dx}(\coth x) = -\text{csch}^2 x$$

$$\frac{d}{dx}(\text{sech}\ x) = -\text{sech} x \tanh x \qquad \frac{d}{dx}(\text{csch}\ x) = -\text{csch} x \coth x$$

$$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{x^2+1}}$$

Antiderivatives.

$$\int x^n\, dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1 \qquad \int \frac{1}{x}\, dx = \ln|x| + C$$

$$\int \sin x\, dx = -\cos x + C \qquad \int \cos x\, dx = \sin x + C$$

$$\int \sec^2 x\, dx = \tan x + C \qquad \int \csc^2 x\, dx = -\cot x + C$$

$$\int \tan x \sec x\, dx = \sec x + C \qquad \int \cot x \csc x\, dx = -\csc x + C$$

$$\int \tan x\, dx = \ln|\sec x| + C \qquad \int \cot x\, dx = \ln|\sin x| + C$$

$$\int e^x\, dx = e^x + C \qquad \int a^x\, dx = \frac{a^x}{\ln a} + C,\ a > 0$$

$$\int \frac{1}{\sqrt{1-x^2}}\, dx = \sin^{-1} x + C \qquad \int \frac{1}{1+x^2}\, dx = \tan^{-1} x + C$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\, dx = \sin^{-1}\left(\frac{x}{a}\right) + C,\ |x| < |a| \qquad \int \frac{1}{x^2+a^2}\, dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{\sqrt{x^2+1}}\, dx = \sinh^{-1} x + C \qquad \int \frac{1}{\sqrt{x^2+a^2}}\, dx = \sinh^{-1}\left(\frac{x}{a}\right) + C$$

① a) $\vec{AB} = \vec{OB} - \vec{OA}$

$= \begin{pmatrix}4\\5\\6\end{pmatrix} - \begin{pmatrix}1\\2\\3\end{pmatrix}$

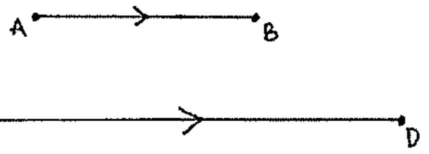
$= \begin{pmatrix}3\\3\\3\end{pmatrix}$

$\vec{CD} = \vec{OD} - \vec{OC}$

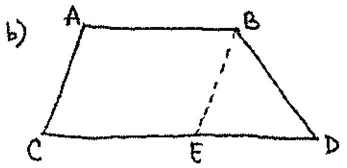
$= \begin{pmatrix}-3\\0\\3\end{pmatrix} - \begin{pmatrix}-9\\-6\\-3\end{pmatrix}$

$= \begin{pmatrix}6\\6\\6\end{pmatrix}$

Since $\vec{CD} = 2\vec{AB}$, therefore $\vec{CD} \parallel \vec{AB}$, and $|\vec{CD}| = 2|\vec{AB}|$.



b)



Draw a dotted line joining point B and E, such that:

- ABEC is a parallelogram,
- $\vec{AB} = \vec{CE}$,
- BED is a triangle

$\vec{AB} = \vec{CE}$

$\begin{pmatrix}3\\3\\3\end{pmatrix} = \vec{OE} - \vec{OC}$

$\vec{OE} = \begin{pmatrix}3\\3\\3\end{pmatrix} + \begin{pmatrix}-9\\-6\\-3\end{pmatrix}$

$= \begin{pmatrix}-6\\-3\\0\end{pmatrix}$

$\vec{CA} = \vec{OA} - \vec{OC}$

$= \begin{pmatrix}1\\2\\3\end{pmatrix} - \begin{pmatrix}-9\\-6\\-3\end{pmatrix}$

$= \begin{pmatrix}10\\8\\6\end{pmatrix}$

$\vec{CE} = \begin{pmatrix}3\\3\\3\end{pmatrix}$ → since $\vec{CE} = \vec{AB}$

$\vec{EB} = \vec{OB} - \vec{OE}$

$= \begin{pmatrix}4\\5\\6\end{pmatrix} - \begin{pmatrix}-6\\-3\\0\end{pmatrix}$

$= \begin{pmatrix}10\\8\\6\end{pmatrix}$

$\vec{ED} = \vec{OD} - \vec{OE}$

$= \begin{pmatrix}-3\\0\\3\end{pmatrix} - \begin{pmatrix}-6\\-3\\0\end{pmatrix}$

$= \begin{pmatrix}3\\3\\3\end{pmatrix}$

Area of ABDC $= |\vec{CA} \times \vec{CE}| + \frac{1}{2}|\vec{EB} \times \vec{ED}|$

$$= \left\| \begin{matrix} i & j & k \\ 10 & 8 & 6 \\ 3 & 3 & 3 \end{matrix} \right\| + \frac{1}{2}\left\| \begin{matrix} i & j & k \\ 10 & 8 & 6 \\ 3 & 3 & 3 \end{matrix} \right\|$$

$$= \frac{3}{2}\left| \left(\begin{vmatrix} 8 & 6 \\ 3 & 3 \end{vmatrix} i - \begin{vmatrix} 10 & 6 \\ 3 & 3 \end{vmatrix} j + \begin{vmatrix} 10 & 8 \\ 3 & 3 \end{vmatrix} k \right) \right|$$

$$= \frac{3}{2}|6i - 12j + 6k|$$

$$= \frac{3}{2}\sqrt{6^2 + (-12)^2 + 6^2}$$

$$= \frac{3}{2} \times 6\sqrt{6}$$

$$= 9\sqrt{6} \text{ units}^2$$

$$\boxed{\therefore\ m = 9,\ n = 6}$$

② a) i.

$$w = \frac{\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)^2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^3}{\left(\cos\frac{\pi}{12} - i\sin\frac{\pi}{12}\right)^2}$$

$$= \frac{\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)^2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^3}{\left(\cos\left(-\frac{\pi}{12}\right) + i\sin\left(-\frac{\pi}{12}\right)\right)^2}$$

$$= \frac{e^{i\left(-\frac{\pi}{2}\right)} \cdot e^{i\pi}}{e^{i\left(-\frac{\pi}{6}\right)}}$$

$$= e^{i\left(\frac{2\pi}{3}\right)}$$

$$\boxed{\begin{aligned}&\therefore |w| = 1\\ &\arg w = \tfrac{2\pi}{3}\end{aligned}}$$

ii. $w = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$

$w^3 = \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^3$, and by De Moivre's Theorem,

$w^3 = \cos 2\pi + i\sin 2\pi$

$w^3 = \underline{\underline{1}}$ $\quad$ ∴ Hence, it is shown that w is a cube root of 1.

b) Let $z = x + iy$

$z + |z+12| = 6 + 6i$

$x + iy + \sqrt{(x+12)^2 + y^2} = 6 + 6i$

Comparing Real & Imaginary parts,

Real : $x + \sqrt{(x+12)^2 + y^2} = 6$

$(x+12)^2 + y^2 = (6-x)^2$

$\cancel{x^2} + 24x + 144 + y^2 = \cancel{x^2} - 12x + 36$

$36x = 36 - 144 - y^2$ -- (1)

Imaginary : $y = 6$ -- (2)

Solving (1) and (2),

$36x = 36 - 144 - 6^2$

$= -144$

$x = -4$

$\boxed{\therefore z = -4 + 6i}$

$-z = 4 - 6i$

$iz = -6 - 4i$

$z^* = \underline{\underline{-4 - 6i}}$

③ a) $f(x) = ax^3 + bx^2 + cx$

$f'(x) = 3ax^2 + 2bx + c$

At the turning points, $f'(x) = 0$

$f'(1) = 3a + 2b + c = \underline{\underline{0}}$

$f'(-2) = 12a - 4b + c = \underline{\underline{0}}$ $\quad$ −shown−

Remaining 2 eqns come from f(x).

$f(1) = a + b + c = \underline{\underline{-7}}$ -- (1)

$f(-2) = -8a + 4b - 2c = \underline{\underline{20}}$ -- (2)

b) By cofactor expansion along the 3rd row,

$$\begin{vmatrix}12&-4&1\\3&2&1\\1&1&1\end{vmatrix} = 1\cdot\begin{vmatrix}-4&1\\2&1\end{vmatrix} - 1\cdot\begin{vmatrix}12&1\\3&1\end{vmatrix} + 1\cdot\begin{vmatrix}12&-4\\3&2\end{vmatrix}$$

$$= -6 - 9 + 36$$

$$= 21$$ -shown-

c) $\begin{cases}12a-4b+c=0\\3a+2b+c=0\\a+b+c=-7\end{cases}$, since $\begin{vmatrix}12&-4&1\\3&2&1\\1&1&1\end{vmatrix} \neq 0$, therefore Cramer's Rule applies.

$$C = \frac{\begin{vmatrix}12&-4&0\\3&2&0\\1&1&-7\end{vmatrix}}{21}$$

use cofactor expansion along the 3rd column

$$= \frac{-7\begin{vmatrix}12&-4\\3&2\end{vmatrix}}{21} = -12 .$$

④ a) $f'(x) = \lim_{y\to x} \dfrac{f(y)-f(x)}{y-x}$

$$= \lim_{y\to x} \frac{y^{2/3}-x^{2/3}}{y-x}$$

$$= \lim_{y\to x} \frac{\cancel{(y^{1/3}-x^{1/3})}(y^{1/3}+x^{1/3})}{\cancel{(y^{1/3}-x^{1/3})}(y^{2/3}+y^{1/3}x^{1/3}+x^{2/3})}$$

$$= \frac{2x^{1/3}}{3x^{2/3}}$$

$$= \frac{2}{3}x^{-1/3} .$$

b) For f to be differentiable, $\lim_{x\to 0} \dfrac{f(x)-f(0)}{x-0}$ must exist.

$$\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} \frac{x^3+x+1-1}{x} = \lim_{x\to 0^+} x^2+1 = 1$$

$$\lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \frac{a\sin x+b-b}{x} = \lim_{x\to 0^-} \frac{a\sin x}{x}$$

$$\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0}$$

$$1 = \lim_{x\to 0^-} \frac{a\sin x}{x}$$ → $\frac{0}{0}$ indeterminate form, LHR applies

$$\lim_{x\to 0^-} \frac{a\cos x}{1} = 1$$

$$a = 1$$

f is differentiable implies f is continuous

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$$

$$\lim_{x\to 0^-} (\sin x + b) = \lim_{x\to 0^+} (x^3+x+1) = 1$$

$$b = 1$$

c) i. $x^{2/3} + y^{2/3} = 4 \quad \frac{d}{dx}$

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-\frac{2}{3}x^{-1/3}}{\frac{2}{3}y^{-1/3}} = -\left(\frac{y}{x}\right)^{1/3}$$

ii. $\left.\frac{dy}{dx}\right|_{x=x_0} = -\left(\frac{y_0}{x_0}\right)^{1/3}$ → gradient of tangent at (x_0, y_0)

Eqn of tangent → $y - y_0 = -\left(\frac{y_0}{x_0}\right)^{1/3}(x - x_0)$

$$\frac{y_0 - y}{y_0^{1/3}} = \frac{x - x_0}{x_0^{1/3}}$$

$$y_0^{2/3} - \frac{y}{y_0^{1/3}} = \frac{x}{x_0^{1/3}} - x_0^{2/3}$$

$$x_0^{2/3} + y_0^{2/3} = \frac{x}{x_0^{1/3}} + \frac{y}{y_0^{1/3}}$$

$$\frac{x}{x_0^{1/3}} + \frac{y}{y_0^{1/3}} = 4$$

-shown-

Since (x_0, y_0) lies on the curve, it fulfills the equation $x_0^{2/3} + y_0^{2/3} = 4$.

⑤ a) $\lim_{x\to 1^+} \frac{(x^2-1)e^x}{\sqrt{x}-1}$

$$= \lim_{x\to 1^+} \frac{(x+1)(x-1)e^x}{\sqrt{x}-1}$$

$$= \lim_{x\to 1^+} \frac{(x+1)\cancel{(\sqrt{x}-1)}(\sqrt{x}+1)e^x}{\cancel{\sqrt{x}-1}}$$

$$= 4e$$

b) Since $-1 \le \sin\frac{1}{x} \le 1$, therefore the denominator of the function must be a positive real number.

And since $\lim_{x\to 0} \sin x = 0$, hence we can conclude

$$\lim_{x\to 0} \frac{\sin x}{\sqrt{x+20} + \sin\frac{1}{x}} = 0$$

c) $\lim_{x\to 0^+} x^{x^2} = \lim_{x\to 0^+} e^{\ln x^{x^2}}$

$$= e^{\lim_{x\to 0^+} \frac{\ln x}{1/x^2}} \rightarrow \frac{\infty}{\infty}, \text{ LHR applies}$$

$$= e^{\lim_{x\to 0^+} \frac{1/x}{-2/x^3}}$$

$$= e^{\lim_{x\to 0^+} -\frac{1}{2}x^2}$$

$$= e^0$$

$$= 1$$

⑥ a) $V = L^3$

$\frac{dV}{dt} = 3L^2 \frac{dL}{dt}$ (take L = 30 cm)

$10 = 3(30)^2 \frac{dL}{dt}$

$\frac{dL}{dt} = \frac{1}{270}$

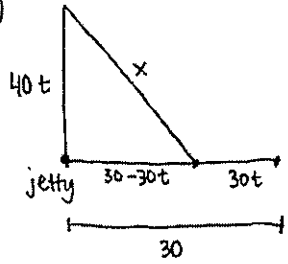
Surface area (S) = $6L^2$

$\frac{dS}{dt} = 12L \frac{dL}{dt}$

$= 12 \times 30 \times \frac{1}{270}$

$= \frac{4}{3}$ cm²/min //.

b)



Find minimum x

$x = \sqrt{(40t)^2 + (30-30t)^2} = \sqrt{2500t^2 - 1800t + 900}$

$\frac{dx}{dt} = \frac{5000t - 1800}{2\sqrt{2500t^2 - 1800t + 900}} = 0$

$t = \frac{18}{50} = 0.36$ hrs

1.00 pm + 0.36 hrs = $\underline{\underline{1.21.36 \text{ pm}}}$.

⑦ a) $\int \frac{x+2}{\sqrt{7-2x^2+4x}}\,dx = \int \frac{x-1}{\sqrt{7-2x^2+4x}}\,dx + 3\int \frac{dx}{\sqrt{7-2x^2+4x}}$

Let $u = 7-2x^2+4x$
$du = (-4x+4)\,dx$

$= \int \frac{x-1}{-4x+4} \frac{du}{\sqrt{u}} + 3\int \frac{dx}{\sqrt{9-(\sqrt{2}x-\sqrt{2})^2}}$

$= -\frac{1}{4}\cdot 2\sqrt{u} + \cancel{3}\int \frac{dx}{\cancel{3}\sqrt{1-\left(\frac{\sqrt{2}}{3}x-\frac{\sqrt{2}}{3}\right)^2}}$

$= -\frac{1}{2}\sqrt{7-2x^2+4x} + \frac{3}{\sqrt{2}}\int \frac{\frac{\sqrt{2}}{3}\,dx}{\sqrt{1-\left(\frac{\sqrt{2}}{3}x-\frac{\sqrt{2}}{3}\right)^2}}$

$= -\frac{1}{2}\sqrt{7-2x^2+4x} + \frac{3}{\sqrt{2}}\sin^{-1}\left(\frac{\sqrt{2}}{3}x-\frac{\sqrt{2}}{3}\right) + C$ //

b) $\int \frac{dx}{e^x+1} = \int\left(1-\frac{e^x}{e^x+1}\right)dx$, Let $u = e^x+1$
$du = e^x\,dx$

$= \int dx - \int \frac{du}{u}$

$= x - \ln|e^x+1| + c$ //

"This suggested solution was done by a student with grade A or above.
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c) $$\lim_{n\to\infty} \frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{n+1}}+\frac{1}{\sqrt{n+2}}+\dots+\frac{1}{\sqrt{n+n}}\right)$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n}\sqrt{\frac{n}{n+k}} \qquad \left(\text{Let } x = \frac{k}{n}\right)$$

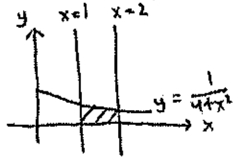
$$= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n}\sqrt{\frac{1}{1+\frac{k}{n}}}$$

$$= \int_0^1 \sqrt{\frac{1}{1+x}}\, dx$$

$$= 2\left[\sqrt{1+x}\right]_0^1$$

$$= 2\sqrt{2} - 2 \quad //$$

⑧ a)



$$\text{Area} = \int_1^2 \frac{dx}{x^2+4}$$

$$= \frac{1}{4}\int_1^2 \frac{dx}{1+\frac{1}{4}x^2}$$

$$= \frac{1}{2}\int_1^2 \frac{\frac{1}{2}\,dx}{1+(\frac{1}{2}x)^2}$$

$$= \frac{1}{2}\left[\tan^{-1}\left(\tfrac{1}{2}x\right)\right]_1^2 = \frac{\pi}{8} - \frac{1}{2}\tan^{-1}\left(\tfrac{1}{2}\right) \quad //$$

b) Use cylindrical shell method

$$\text{volume} = 2\pi \int_1^2 \underbrace{(x-1)}_{\text{radius}} \underbrace{\frac{1}{x^2+4}}_{\text{height}}\, dx$$

$$= 2\pi\left(\int_1^2 \frac{x}{x^2+4}\,dx - \int_1^2 \frac{dx}{x^2+4}\right), \quad \text{Let } u = x^2+4, \ du = 2x\,dx, \quad \text{when } x=1,\ u=5;\ x=2,\ u=8$$

$$= \pi\int_5^8 \frac{du}{u} - 2\pi\left(\frac{\pi}{8} - \frac{1}{2}\tan^{-1}\left(\tfrac{1}{2}\right)\right)$$

$$= \pi \ln\frac{8}{5} - \frac{\pi^2}{4} + \pi\tan^{-1}\left(\tfrac{1}{2}\right) \quad //$$

NANYANG TECHNOLOGICAL UNIVERSITY
SEMESTER I EXAMINATION 2020–2021
MH1810 – Mathematics 1

NOVEMBER 2020 TIME ALLOWED: 2 HOURS

Matriculation Number:

Seat Number:

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **NINE (9)** questions and comprises **EIGHTTEEN (18)** pages, including an Appendix.
2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
3. This **IS NOT** an **OPEN BOOK** exam. However, a list of formulae is provided in Appendix Pages 16-18.
4. Candidates may use calculators. However, they should write down systematically the steps in the workings.
5. All your solutions should be written in this booklet within the space provided after each question. If you use an additional answer book, attach it to this booklet and hand them in at the end of the examination.
6. This examination paper is **NOT ALLOWED** to be removed from the examination hall.

For examiners only

Questions	Marks	Questions	Marks	Questions	Marks
1 (10)		4 (10)		7 (15)	
2 (10)		5 (10)		8 (15)	
3 (10)		6 (10)		9 (10)	

Total (100)	

QUESTION 1. **(10 Marks)**

The points $A(1,2,3)$, $B\,(1,2,4)$ and $C\,(1,3,5)$ are shown in Figure 1.

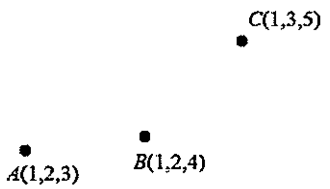


Figure 1.

(a) If D is a point such that $\overrightarrow{AD} = \left(\overrightarrow{AC}\cdot\overrightarrow{AB}\right)\overrightarrow{AB}$, find the coordinates of D. Indicate point D in Figure 1.

Question 1 continues on Page 3.

(b) Find

(i) the area of triangle ABC, and

(ii) the angle $\angle CAD$.

QUESTION 2. **(10 Marks)**

(a) Find all solutions of the equation $\overline{(z^4)} + 324 = 0$. Express your answers in the form $x + iy$. Indicate the solutions on the Argand diagram below.

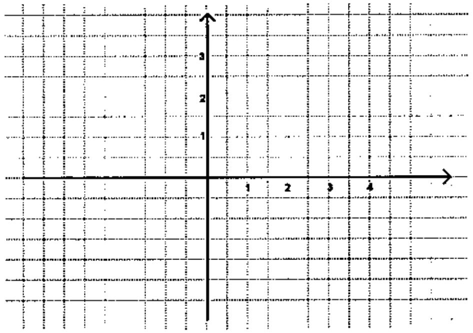


Figure 2.

Question 2 continues on Page 5.

(b) Solve the equation

$$|z-3| = z - i.$$

Express your answer z in the form $x + iy$.

QUESTION 3. **(10 Marks)**

(a) Find the determinant of the matrix $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & a & 1 \end{pmatrix}$, in terms of the unknown constant a.

(b) Use part (a) and Cramer's Rule to find the value of x that satisfies the system of linear equations

$$\begin{aligned} \frac{2}{x} + \frac{1}{y} + z &= 2, \\ \frac{1}{x} + \frac{1}{y} + z &= 1, \\ -\frac{1}{y} + z &= 0. \end{aligned}$$

QUESTION 4. **(10 Marks)**

Evaluate the following limits.

(a) $\lim_{x \to \infty} \frac{x}{x^2+1} e^{\sin x}$.

(b) $\lim_{x \to 1^-} \frac{\sqrt[3]{x}-1}{x^2-1} \sin\left(\frac{\pi}{1+x}\right)$.

QUESTION 5. **(10 Marks)**

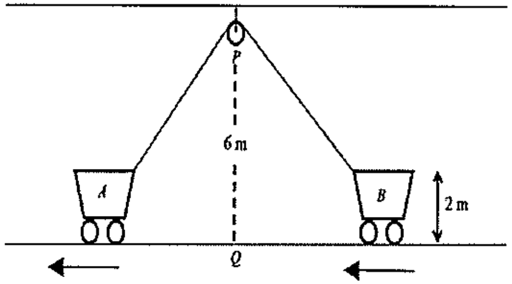
(a) Let $f(x) > 0$ be a differentiable function on $\mathbf{R}$. Use the **definition of derivative** to prove that

$$\frac{d}{dx}\sqrt{f(x)} = \frac{1}{2}\frac{f'(x)}{\sqrt{f(x)}}.$$

(b) Find the equation of the tangent to the graph of $y = x^x$ at $x = 2$.

QUESTION 6 **(10 Marks)**

A rope of length 12 m that passes over a pulley at P is connected to the top corners of two identical carts A and B of height 2 m. The point Q is on the floor 6 m directly below P. Cart A is being pulled away from Q at a speed of 0.5 m/s. How fast is Cart B moving towards Q when Cart A is 3 m from Q?



QUESTION 7. **(15 Marks)**

(a) Determine the dimensions of the rectangle of largest possible area that can be inscribed in a semicircle of radius 3 cm.

Question 7 continues on Page 11.

(b) Show that the equation $x^3 + 3x + 1 = 0$ has exactly one real solution.

QUESTION 8. **(15 Marks)**

(a) Find the values of p such that the improper integral $\int_0^1 x^p \ln x \, dx$ converges. For those values of p such that the integral converges, evaluate the value of the integral.

Question 8 continues on Page 13.

(b) Let

$$f(x) = \begin{cases} x\left(|\ln x|\right)^{1/2}, & x \neq 0, \\ 0, & x = 0, \end{cases}$$

and R be the region bounded by the curve of $y = f(x)$, x-axis, y-axis and $x = 1$. Find the volume of the solid obtained by rotating R about the x-axis.

QUESTION 9. **(10 Marks)**

Evaluate the following integrals.

(a) $\displaystyle\int \frac{1}{(x+2)(x+3)}dx.$

Question 9 continues on Page 15.

(b) $\int \frac{1}{x^2+x+1}\,dx.$

END OF PAPER

Appendix

Numerical Methods.

- Linearization Formula:

$$L(x) = f(a) + f'(a)\,(x-a)$$

- Newton's Method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

- Trapezoidal Rule:

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\left[y_0 + 2\left(y_1 + y_2 + \cdots + y_{n-1}\right) + y_n\right]$$

- Simpson's Rule:

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\left[y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n\right]$$

Derivatives.

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(e^x) = e^x \qquad \frac{d}{dx}(a^x) = a^x \ln a$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x} \qquad \frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\cosh x) = \sinh x$$

$$\frac{d}{dx}(\tanh x) = \mathrm{sech}^2 x \qquad \frac{d}{dx}(\coth x) = -\mathrm{csch}^2 x$$

$$\frac{d}{dx}(\mathrm{sech}\ x) = -\mathrm{sech} x \tanh x \qquad \frac{d}{dx}(\mathrm{csch}\ x) = -\mathrm{csch} x \coth x$$

$$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{x^2+1}}$$

Antiderivatives.

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1 \qquad \int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \csc^2 x\,dx = -\cot x + C$$

$$\int \tan x \sec x\,dx = \sec x + C \qquad \int \cot x \csc x\,dx = -\csc x + C$$

$$\int \tan x\,dx = \ln|\sec x| + C \qquad \int \cot x\,dx = \ln|\sin x| + C$$

$$\int e^x\,dx = e^x + C \qquad \int a^x\,dx = \frac{a^x}{\ln a} + C,\ a > 0$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C \qquad \int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C,\ |x| < |a| \qquad \int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{\sqrt{x^2+1}}\,dx = \sinh^{-1} x + C \qquad \int \frac{1}{\sqrt{x^2+a^2}}\,dx = \sinh^{-1}\left(\frac{x}{a}\right) + C$$

Subject Code	: MH1810
Subject Name	: Mathematics 1
Year / Semester	: 20-21/1

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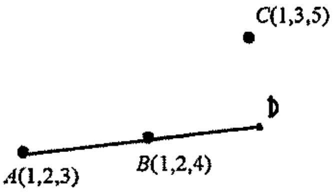

Figure 1.

1. $\overrightarrow{AD} = (\overrightarrow{AC} \cdot \overrightarrow{AB}) \cdot \overrightarrow{AB}$

$\overrightarrow{AD} = \left(\left[\begin{pmatrix}1\\3\\5\end{pmatrix} - \begin{pmatrix}1\\2\\3\end{pmatrix} \right] \cdot \left[\begin{pmatrix}1\\2\\4\end{pmatrix} - \begin{pmatrix}1\\2\\3\end{pmatrix} \right] \right) \cdot \overrightarrow{AB}$

$= \left[\begin{pmatrix}0\\1\\2\end{pmatrix} \cdot \begin{pmatrix}0\\0\\1\end{pmatrix} \right] \cdot \begin{pmatrix}0\\0\\1\end{pmatrix}$

$= \begin{pmatrix}0\\0\\2\end{pmatrix}$

$\overrightarrow{AD} = \begin{pmatrix}0\\0\\2\end{pmatrix} = \vec{d} - \vec{a}$

$= \vec{d} - \begin{pmatrix}1\\2\\3\end{pmatrix}$

$\vec{d} = \begin{pmatrix}1\\2\\5\end{pmatrix}$

$\overrightarrow{AD} = \begin{pmatrix}0\\0\\2\end{pmatrix} = 2\overrightarrow{AB}$, $\overrightarrow{AD} \parallel \overrightarrow{AB}$

b. $\text{area} = \frac{1}{2} \| \overrightarrow{AC} \times \overrightarrow{AB} \| = \frac{1}{2} \times \left\| \begin{pmatrix}0\\1\\2\end{pmatrix} \times \begin{pmatrix}0\\0\\1\end{pmatrix} \right\|$

$= \frac{1}{2}$

ii) $\overrightarrow{AC} \cdot \overrightarrow{AD} = \|\overrightarrow{AC}\| \, \|\overrightarrow{AD}\| \cos\theta$

$\begin{pmatrix}0\\1\\2\end{pmatrix} \cdot \begin{pmatrix}0\\0\\2\end{pmatrix} = \sqrt{5}\sqrt{4} \cos\theta$

$\cos\theta = \frac{2}{5}\sqrt{5}$

$\theta \approx 26.565°$

2a.

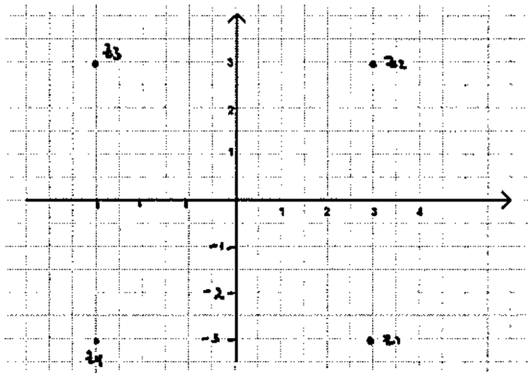

Figure 2.

$z^4 = 324(-1+0i)$

$z^4 = 324(e^{\pi i + 2\pi k})^{1/4}$

$z = 3(e^{\pi/4 i + \pi/2 k})$

at $k=0$, $z = 3e^{-\pi/4 i} = 3-3i$

$k=1$, $z = 3e^{\pi/4 i} = 3+3i$

$k=2$, $z = 3e^{3\pi/4 i} = -3+3i$

$k=3$, $z = 3e^{5\pi/4 i} = -3-3i$

b. $|z-3| = z-i$

assume $z = a+bi$, since $|z-3|$ must be real,

$z-i = a+i(b-1) = a+0i$

$b=1$

$z = a+i$

$|z-3| = z-i$

$|a+i-3| = (a+i)-i$

$\sqrt{(a-3)^2+1^2} = a$

$a^2-6a+9+1 = a^2$

$10 = 6a$; $\underline{a = 5/3}$ $\underline{z = 5/3 + i}$

"This suggested solution was done by a student with grade A or above. MSE Club specifically disclaims any responsibility for any errors in the answers given. Caveat lector"

Subject Code : MH1810

Subject Name : Mathematics 1

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3a. $\det A = 2\begin{vmatrix} 1 & 1 \\ a & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ a & 1 \end{vmatrix}$

$= 2(1-a) - 1(1-a) = 1-a$

b. let $1/x = a$, $1/y = b$, $z = c$

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

let matrix $\begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix} = M$

$\det M = 1-(-1) = 2$

$$a = \frac{\det \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}}{\det M} = 1 \longrightarrow x = 1$$

4a. $-1 \leq \sin x < 1$

$$\lim_{x\to\infty} e^{-1} \leq \lim_{x\to\infty} e^{\sin x} \leq \lim_{x\to\infty} e^{1}$$

$$\lim_{x\to\infty} \frac{x}{x^2+1} e^{-1} \leq \lim_{x\to\infty} \frac{x}{x^2+1} e^{\sin x} \leq \lim_{x\to\infty} \frac{x}{x^2+1} e^{1}$$

$$\lim_{x\to\infty} \frac{1/x}{1+1/x^2} e^{-1} \leq \lim_{x\to\infty} \frac{1/x}{1+1/x^2} e^{\sin x} \leq \lim_{x\to\infty} \frac{1/x}{1+1/x^2} e^{1}$$

since $\lim_{x\to\infty} \frac{1/x}{1+1/x^2} e^{-1} = \lim_{x\to\infty} \frac{1/x}{1+1/x^2} e^{1} = 0$,

By squeeze theorem, $\lim_{x\to\infty} \frac{x}{x^2+1} e^{\sin x} = \underline{0}$.

b) $\lim_{x\to 1^-} \frac{\sqrt[3]{x}-1}{x^2-1} \sin\left(\frac{\pi}{1+x}\right) =$

$\lim_{x\to 1^-} \frac{\cancel{\sqrt[3]{x}-1}}{(x+1)\cancel{(\sqrt[3]{x}-1)}(\sqrt[3]{x^2}+\sqrt[3]{x}+1)} \sin\left(\frac{\pi}{1+x}\right) =$

$1/6 \cdot \sin(\pi/2) = \underline{1/6}$.

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5. let $g(x) = \sqrt{f(x)}$

$$\frac{d}{dx}\, g(x) = \lim_{x \to c} \frac{g(x) - g(c)}{x-c}$$

$$= \lim_{x \to c} \frac{\sqrt{f(x)} - \sqrt{f(c)}}{x-c} \cdot \frac{\sqrt{f(x)} + \sqrt{f(c)}}{\sqrt{f(x)} + \sqrt{f(c)}}$$

$$= \lim_{x \to c} \frac{f(x) - f(c)}{x-c} \cdot \frac{1}{\sqrt{f(x)} + \sqrt{f(c)}}$$

$$= f'(x) \cdot {}^{1}/_{2\sqrt{f(x)}} = \frac{1}{2} \frac{f'(x)}{\cdot \sqrt{f(x)}} \quad \text{(PROVEN)}$$

6. $y = x^x$

$\underline{\ln y = x \ln x} \times {}^{d}/_{dx}$

${}^{1}/_{y}\ {}^{dy}/_{dx} = \ln x + x \cdot {}^{1}/_{x}$

${}^{dy}/_{dx} = y\,(\ln x + 1)$

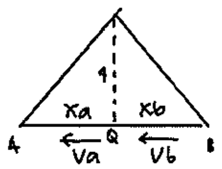
at $x = 2$, $y = 4$, ${}^{dy}/_{dx} = 4(\ln 2 + 1)$

$\therefore\ y - y_1 = m(x - x_1)$

$y - 4 = 4(\ln 2 + 1)\,(x - 2)$

$\underline{y = (4\ln 2 + 4)x - (8\ln 2 + 4)}$,

6.



let Q = (0,0)

$$\frac{dx_a}{dt} = -0.5\,{}^{m}/_{s}$$

$$\sqrt{x_a^2 + 4^2} + \sqrt{x_b^2 + 4^2} = 12$$

When $x_a = -3$

$5 + \sqrt{x_b^2 + 16} = 12$

$x_b^2 + 16 = 49 \quad x_b = \pm\sqrt{33}$

$\therefore$ B on A's right side, so $x_b = +\sqrt{33}$ ... (1)

$$\frac{d}{dt}\left(\sqrt{x_a^2+4^2} + \sqrt{x_b^2+4^2}\right) = 0$$

$${}^{d}/_{dx_a} \cdot (x_a^2+16)^{1/2}\ {}^{dx_a}/_{dt} + {}^{d}/_{dx_b} \cdot (x_b^2+16)\ {}^{dx_b}/_{dt} = 0$$

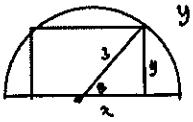
$$\frac{x_a}{\sqrt{x_a^2+16}}(-0.5) + \frac{x_b}{\sqrt{x_b^2+16}}\ \frac{dx_b}{dt} = 0$$

$$\frac{dx_b}{dt} = \frac{x_a}{2\sqrt{x_a^2+16}} \cdot \frac{\sqrt{x_b^2+16}}{x_b}$$

$$= \frac{-3}{2\sqrt{25}} \cdot \frac{\sqrt{49}}{\sqrt{33}} = \frac{-21}{10\sqrt{33}}$$

speed of B $= \left\| \frac{-21}{10\sqrt{33}} \right\| \approx \underline{0.36556\ {}^{m}/_{s}}$ //

7.

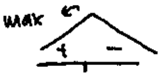


$y = 3\sin\theta \quad x = 3\cos\theta$

area $= 2x \cdot y$

$= 6\cos\theta \cdot 3\sin\theta$

$= 18\sin\theta\cos\theta = 9\sin 2\theta$



at $\theta = 0$

$\frac{dA}{dt} = 0$

$\frac{dA}{d\theta} = 0$

$9\cos 2\theta \cdot 2 = 0$

$\cos 2\theta = 0$

$2\theta = \pm\pi/2 + 2\pi k$

$\theta = \pm\pi/4 + \pi k$

$\theta = \pi/4$

$y = 3\sin\pi/4 \quad x = 3\cos\pi/4$

rectangle with length $= 2x = 6(\frac{1}{2}\sqrt{2}) = 3\sqrt{2}$

width $= y = \frac{3}{2}\sqrt{2}$

7b. $x^3 + 3x + 1 = 0 \longrightarrow f(x)$

- $f(-1) = -3$, $f(1) = 5$

  $f(-1) < 0 < f(1)$

  by IVT, $\exists\, c \in (-1, 1)$ such that $f(c) = ? = 0$
- since $f(x)$ is polynomials, $f(x)$ is continuous on $[-1, 1]$ and differentiable on $(-1, 1)$. Rolle's theorem applies.

  we assume there's more than one real root which is d, such that $f(c) = f(d) = 0$

  by Rolle's Theorem, $\exists\, m \in (c, d)$ such that $f'(m) = 0$

  if $f(m) = m^3 + 3m + 1$,

  $f'(m) = 3m^2 + 2$, which is always positive (cannot be 0)

  Therefore by method of contradiction, equation $x^3 + 3x + 1 = 0$ has exactly one real solution.

Subject Code : MH1810

Subject Name : Mathematics 1

Year / Semester : 20-21/1

Suggested Solution Brought to You by Materials Science and Engineering Club

8a. $\int_0^1 x^p \ln x \, dx = \lim_{t\to 0} \int_t^1 x^p \ln x \, dx$

let $u = \ln x$ $\quad dv = x^p$

$du = \frac{1}{x} dx \quad v = \frac{1}{p+1} x^{p+1}$

$= \frac{x}{p+1} \ln x \Big|_0^1 - \lim_{t\to 0} \int_t^1 \frac{x^{p+1}}{p+1} \frac{1}{x} dx$

$= 0 - \lim_{t\to 0} \int_t^1 \frac{x^{p+1}}{p+1} 1/x \, dx$

$= -\frac{1}{(p+1)^2} x^{p+1} \Big|_0^1$

$p > -1 \to \frac{x^{(p+1)}}{p+1} \ln x - \frac{1}{(p+1)^2} \longrightarrow$ converge

$p \in -1 \longrightarrow x^{p+1} \Big|_0^1$ is not defined $\longrightarrow$ diverse

the value p : $p > -1$

value of integral : $\frac{x^{p+1}}{p+1} \ln x \Big|_0^1 - \frac{1}{(p+1)^2}$

b. $\lim_{x\to 0} x(|\ln x|)^{1/2} = \lim_{x\to 0} \sqrt{\left(\frac{(\ln x)^{1/2}}{1/x}\right)^2}$

$= \sqrt{\lim_{x\to 0} \ln x / 1/x^2}$

$= \sqrt{\lim_{x\to 0} 1/x / -3 1/x^3}$

$= \sqrt{\lim_{x\to 0} (-1/3 x^2)} = 0$

since $\lim_{x\to 0} x(|\ln x|)^{1/2} = f(0)$, f is continuous at 0,

and since f is continuous until $x=1$,

volume is $\left|\pi \int_0^1 (x(|\ln x|)^{1/2})^2 dx\right| = \left|\pi \int_0^1 x^2 \ln x \, dx\right|$

$= \left|\pi \left(-\frac{x}{(p+1)^2} \Big|_0^1\right)\right|$

$= |\pi(-1/9)| = \underline{\pi/9 \text{ units}^2}$

using solution in 8a

9a. $\frac{A}{(x+2)} + \frac{B}{(x+3)} = \frac{A(x+3) + B(x+2)}{(x+2)(x+3)}$

$A(x+3) + B(x+2) = 1$

$Ax + Bx + 3A + 2B = 1$

$A = 1 \quad B = -1$

$\int \frac{1}{(x+2)} dx + \int \frac{-1}{(x+3)} dx = \underline{\ln|x+2| - \ln|x+3| + c}$

b. $\int \frac{1}{x^2+x+1} dx = \int \frac{1}{(x^2+x+1/4) + 3/4} dx$

$= \int \frac{1}{(x+1/2)^2 + 3/4} dx$

$= \frac{1}{\sqrt{3/4}} \tan^{-1}\left(\frac{x+1/2}{\sqrt{3/4}}\right) + c$

$= \underline{\frac{2}{3}\sqrt{3} \tan^{-1}\left(\frac{2}{3}\sqrt{3}(x+1/2)\right) + c}$