NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER 2 EXAMINATION 2016-2017

MA2006 – ENGINEERING MATHEMATICS

April/May 2017 Time Allowed: $2^1/_2$ hours

INSTRUCTIONS

1. This paper contains **FIVE (5)** questions and comprises **SIX (6)** pages.
2. Answer **ALL** questions.
3. All questions carry equal marks.
4. This is a **CLOSED BOOK** examination.
5. Mathematical tables and formulae are provided on pages 4 to 6.

---

1 (a) Consider the following system of linear algebraic equations:

$$\begin{bmatrix} 2 & -3 & 3 \\ 6 & -8 & 7 \\ -2 & 6 & u \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \\ v \end{bmatrix},$$ where u and v are unknown real constants.

Find the values of u and v such that the system has
(i) A unique solution.
(ii) Many solutions.
(iii) No solution.

(10 marks)

(b) Find the eigenvalues and eigenvectors for the following matrix $\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 0 & 5 \\ 2 & 0 & 1 \end{bmatrix}$ and construct a 3×3 invertible matrix **P** and a 3×3 diagonal matrix **D** such that $\mathbf{A} = \mathbf{PDP}^{-1}$. (Note: no need to find $\mathbf{P}^{-1}$)

(10 marks)

2 (a) If **C** is a N×N matrix such that $\mathbf{C}^3 = \mathbf{0}$ (a matrix whose all elements are zero), prove the inverse of $\mathrm{I} - \mathrm{C}$ is given by $\mathrm{I} + \mathrm{C} + \mathrm{C}^2$. Here I is the N×N identity matrix.

(6 marks)

(b) For the given vector function $\mathbf{F} = (x^3 + 6y)\,\mathbf{i} + (y^2 + zy)\,\mathbf{j} + (x^2 - z)\,\mathbf{k}$, calculate curl (F) and grad(div(F)).

(6 marks)

(c) Assume that the pressure distribution in space is given by $p(x, y, z) = 50 + x^2y^2z^2 + 2y^2$, where x, y and z are the Cartesian coordinates of a point in space. Let the point (1, 1, 3) in space denoted by Q.

(i) At the point Q, calculate the rate of change of the pressure per unit distance in the direction of the vector $2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$.

(4 marks)

(ii) At the point Q, find the direction along which the rate of change of the pressure per unit distance is the maximum. Give this direction in terms of a unit vector.

(4 marks)

3. The surface Σ is the portion of the plane $z = y + 3$ that lies inside the cylindrical region $x^2 + y^2 \le 1$.

(a) Give a unit normal vector to the surface Σ.

(2 marks)

(b) Sketch the surface Σ. Indicate the coordinates of the center of Σ in your sketch. (Note: Let the x axis point out of the paper towards you, the y axis be the horizontal axis pointing towards the right and z axis be vertical.)

(5 marks)

(c) The surface Σ is projected perpendicularly (along the z axis) onto the region D on the Oxy plane ($z = 0$ plane).

(i) Describe the region D.

(1 mark)

(ii) If the areas of infinitesimal parts of Σ and D are given by $d\sigma$ and dA respectively, express $d\sigma$ in terms of dA.

(2 marks)

(iii) Compute the surface integral $\iint_{\Sigma} yz\, d\sigma$. (**Hint.** You may use the formula $\cos(2\theta) = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$ without proving it.)

(10 marks)

18

4. A periodic function is defined in one period interval as $f(x) = \sin x, \ -\frac{\pi}{2} \le x < 0$.

(a) Sketch $f(x)$ over the interval $-\pi \le x < \pi$. (5 marks)

(b) Find the Fourier series for $f(x)$. (10 marks)

(c) Find the value of the Fourier series at $x = \pi$. (5 marks)

5. Laplace Transform

(a) Find the Laplace transform of the function $f(t)$ defined by

$$f(t) = \begin{cases} e^{-2t}\sin(t), & 0 \le t < \pi; \\ 0, & t > \pi. \end{cases}$$

(5 marks)

(b) Find the inverse Laplace transform

$$L^{-1}\left\{\frac{e^{-2s}}{s^2+4s+25}\right\}.$$

(5 marks)

(c) Solve the following ordinary differential equation for $y(t)$ by using the Laplace transform.

$$y' - y = te^{-t} \quad y(0) = 0.$$

(10 marks)

FORMULAE FOR VECTOR CALCULUS PART

1. $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}, \quad \vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k},$

$\vec{a} \bullet \vec{b} = a_1b_1 + a_2b_2 + a_3b_3,$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - b_2a_3)\vec{i} + (a_3b_1 - b_3a_1)\vec{j} + (a_1b_2 - b_1a_2)\vec{k}.$$

2. $grad\ f = \nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$, for scalar function $f(x,y,z)$.

3. $\vec{V} = p(x,y,z)\vec{i} + q(x,y,z)\vec{j} + r(x,y,z)\vec{k},$

$\nabla \bullet \vec{V} = div\vec{V} = \frac{\partial p}{\partial x} + \frac{\partial q}{\partial y} + \frac{\partial r}{\partial z},$

$$\nabla \times \vec{V} = curl\,\vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ p & q & r \end{vmatrix} = \left(\frac{\partial r}{\partial y} - \frac{\partial q}{\partial z}\right)\vec{i} + \left(\frac{\partial p}{\partial z} - \frac{\partial r}{\partial x}\right)\vec{j} + \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right)\vec{k},$$

4. $ds = \sqrt{(dx)^2 + (dy)^2 + (dz)^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt,$

$$\int_C f(x,y,z)ds = \int_{t_1}^{t_2} f[x(t),y(t),z(t)]\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt.$$

5. $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}, \quad d\vec{r} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k},$

For $\vec{F} = p(x,y,z)\vec{i} + q(x,y,z)\vec{j} + r(x,y,z)\vec{k},$

$$\int_C \vec{F} \bullet d\vec{r} = \int p\,dx + q\,dy + r\,dz = \int_{t_1}^{t_2}\left(p\frac{dx}{dt} + q\frac{dy}{dt} + r\frac{dz}{dt}\right)dt.$$

6. Green's Theorem

$$\oint_C f(x,y)dx + g(x,y)dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)dA$$

7. surface integral:

$$\iint_S g(x,y,z)d\sigma = \iint_R g[x,y,z(x,y)]\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\,dA,$$

for the surface given by $z = f(x,y)$

FORMULAE FOR SPECIAL FUNCTIONS

$\sin(x\pm y)=\sin x\cos y\pm\cos x\sin y$ $\quad$ $\sin x\sin y=[-\cos(x+y)+\cos(x-y)]/2$

$\cos(x\pm y)=\cos x\cos y\mp\sin x\sin y$ $\quad$ $\cos x\cos y=[\cos(x+y)+\cos(x-y)]/2$

$\sin x\cos y=[\sin(x+y)+\sin(x-y)]/2$

$\sin^2 x=(1-\cos 2x)/2,\quad \cos^2 x=(1+\cos 2x)/2$

$\sin x+\sin y=2\sin\frac{x+y}{2}\cos\frac{x-y}{2},\quad \sin x-\sin y=2\cos\frac{x+y}{2}\sin\frac{x-y}{2}$

$\cos x+\cos y=2\cos\frac{x+y}{2}\cos\frac{x-y}{2},\quad \cos x-\cos y=-2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$

$\sinh x=(e^x-e^{-x})/2,\quad \cosh x=(e^x+e^{-x})/2$

$\sinh x=-i\sin ix,\quad \cosh x=\cos ix$

FORMULAE FOR FOURIER SERIES AND TRANSFORM

Euler Formulae for Fourier series for a periodic function $f(x)$ with a period $P=2L$

$$f(x)=a_0+\sum_{n=1}^{\infty}(a_n\cos\frac{n\pi x}{L}+b_n\sin\frac{n\pi x}{L})$$

where $a_0=\frac{1}{2L}\int_{-L}^{L}f(x)dx,\ a_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos\frac{n\pi x}{L}dx,\ b_n=\frac{1}{L}\int_{-L}^{L}f(x)\sin\frac{n\pi x}{L}dx$

Complex form of Fourier series for $f(x)$

$$f(x)=\sum_{n=-\infty}^{\infty}c_n e^{in\pi x/L}\quad\text{where}\quad c_n=\frac{1}{2L}\int_{-L}^{L}f(x)e^{-in\pi x/L}dx,\ n=0,\pm1,\pm2,\cdots$$

Fourier integral

$$f(x)=\int_0^{\infty}[A(\omega)\cos\omega x+B(\omega)\sin\omega x]d\omega$$

where $A(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty}f(v)\cos\omega v\,dv$ and $B(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty}f(v)\sin\omega v\,dv$

Complex form of Fourier integral

$$f(x)=\int_{-\infty}^{\infty}C(\omega)e^{i\omega x}d\omega,\qquad C(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}f(x)e^{-i\omega x}dx$$

Cosine transform

$$\hat{f}_c(\omega)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}f(x)\cos(\omega x)dx,\qquad f(x)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}\hat{f}_c(\omega)\cos(\omega x)d\omega$$

Sine transform

$$\hat{f}_s(\omega)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}f(x)\sin(\omega x)dx,\qquad f(x)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}\hat{f}_s(\omega)\sin(\omega x)d\omega$$

Fourier transform

$$\hat{f}(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{-i\omega x}dx,\qquad f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{i\omega x}d\omega$$

LAPLACE TRANSFORM TABLE

$f(t)=L^{-1}\{F(s)\}$	$F(s)=L\{f(t)\}$	$f(t)=L^{-1}\{F(s)\}$	$F(s)=L\{f(t)\}$
1	$\frac{1}{s}$	$\frac{\sinh at}{a}$	$\frac{1}{s^2-a^2}$
$\frac{t^{n-1}}{(n-1)!}$	$\frac{1}{s^n},\ (n=1,2,\cdots)$	$\cosh at$	$\frac{s}{s^2-a^2}$
$\frac{t^{n-1}e^{at}}{(n-1)!}$	$\frac{1}{(s-a)^n},\ (n=1,2,\cdots)$	$u(t-a)$, Unit step function	$\frac{e^{-as}}{s}$
$\frac{\sin\omega t}{\omega}$	$\frac{1}{s^2+\omega^2}$	$\delta(t-a)$, Unit impulse function	e^{-as}
$\cos\omega t$	$\frac{s}{s^2+\omega^2}$		

LAPLACE TRANSFORM FORMULAE

$f(t)=L^{-1}\{F(s)\}$	$F(s)=L\{f(t)\}$	Remarks
$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-\cdots-f^{(n-1)}(0)$	Differentiation of function
$\int_0^t f(\tau)d\tau$	$F(s)/s$	Integration of function
$e^{at}f(t)$	$F(s-a)$	Shift on s-axis
$u(t-a)f(t-a)$	$e^{-as}F(s)$	Shift on t-axis
$tf(t)$	$-F'(s)=-dF/ds$	Differentiation of transform
$t^nf(t)$	$(-1)^nF^{(n)}(s)$	Differentiation of transform
$f(t)/t$	$\int_s^{\infty}F(\tilde{s})d\tilde{s}$	Integration of transform
$\int_0^t f(t-\tau)g(\tau)d\tau = \int_0^t f(\tau)g(t-\tau)d\tau$	$L(f*g)=L(f)L(g)$	Convolution
$f(t)=f(t+p)$	$\int_0^p e^{-st}f(t)dt\Big/(1-e^{-sp})$	$f(t)$ is periodic with period p

END OF PAPER

19

By using row operations

$$\left[\begin{array}{ccc|c} 2 & -3 & 3 & -2 \\ 0 & -1 & 2 & -4 \\ 0 & 3 & 3+u & v-2 \end{array}\right] \begin{array}{l} 3R_1-R_2 \\ R_1+R_3 \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & -3 & 3 & -2 \\ 0 & -1 & 2 & -4 \\ 0 & 0 & 9+u & v-14 \end{array}\right] \quad 3R_2+R_3$$

$9+u = v-14$

When $u=-9$ & $v=14$ ∴ Multiple sol^n

When $u=-9$ & $v\neq 14$] ∴ No sol^n
OR $u\neq -9$ & $v=14$]

When $u\neq -9$ & $v\neq 14$ → ~~Multiple~~ Unique sol^n of the system

b) $$\det\begin{bmatrix} 0-\lambda & 0 & 1 \\ 2 & 0-\lambda & 5 \\ 2 & 0 & 1-\lambda \end{bmatrix} = 0$$

$-\lambda[-\lambda(1-\lambda)] - 0 + [-2(-\lambda)] = 0$

$-\lambda[-\lambda+\lambda^2] + 2\lambda = 0$

$\lambda[2-(-\lambda+\lambda^2)] = 0$

Eigen values: $\lambda=0$, $2+\lambda-\lambda^2=0$

$\lambda=2$ & $\lambda=-1$

When $\lambda=0$,

$$\left[\begin{array}{ccc|c} x & y & z & \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 5 & 0 \\ 2 & 0 & 1 & 0 \end{array}\right]$$

$z=0$

$2x+5z=0$

$2x+z=0$

Let $y=t$

Eigen vector $\begin{pmatrix} 0 \\ t \\ 0 \end{pmatrix}$ //

? P.(i) //

When $\lambda=2$,

x	y	z	
-2	0	1	0
2	-2	5	0
2	0	-1	0

$-2x+z=0$
$2x-2y+5z=0$
$2x-z=0$

Solving them, $z=2x$ & $y=6x$
Let $z=2$, $x=1$, $y=6$

Eigen vector : $\begin{pmatrix}1\\6\\2\end{pmatrix}$

When $\lambda=-1$

x	y	z	
1	0	1	0
2	1	5	0
2	0	2	0

$x+z=0$
$2x+y+5z=0$
$2x+2z=0$

Solving them, $y=-7x$, $x=z$
Let $x=1$, $z=1$, $y=-7$

Eigen vector : $\begin{pmatrix}1\\-7\\1\end{pmatrix}$

$$D=\begin{pmatrix}0&0&0\\0&2&0\\0&0&-1\end{pmatrix}$$

$$P=\begin{pmatrix}0&1&1\\1&6&-7\\0&2&1\end{pmatrix}$$

2a. To prove $A^{-1}=B$, try $AB=I$, else it is not the inverse. $\Rightarrow AA^{-1}=I$

① Identify your A & B first.

② Based on the Qn, ~~I-C is the inverse of~~ the inverse of $I+C+C^2$ is $I-C$

③ So to prove if those 2 are truly inverse, multiply both of them to see if you can get I.

$$(I-C)(I+C+C^2) = I+C+C^2-C-C^2-C^3$$
$$= I-C^3$$

④ READ THE Qn $\Rightarrow$ Since $C^3=0$

$$I-C^3 = I-0$$
$$=I$$

~~∴ LHS = RHS (Shown)~~

b. $\vec{F} = \begin{pmatrix} x^3 + 6y \\ y^2 + zy \\ x^2 - z \end{pmatrix}$

$\text{curl } \vec{F} = \nabla \times \vec{F}$

$$= \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \times \begin{pmatrix} x^3 + 6y \\ y^2 + zy \\ x^2 - z \end{pmatrix} = \begin{pmatrix} 0 - y \\ -(2x - 0) \\ 0 - 6 \end{pmatrix} = \begin{pmatrix} -y \\ -2x \\ -6 \end{pmatrix} //$$

$\text{div } \vec{F} = \nabla \cdot \vec{F}$

$$= \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \cdot \begin{pmatrix} x^3 + 6y \\ y^2 + 2y \\ x^2 - z \end{pmatrix} = 3x^2 + 2y + z - 1$$

$\text{grad}(\text{div } \vec{F}) = \nabla(3x^2 + 2y + z - 1)$

$$= \left(\frac{\partial}{\partial x} \text{div}\vec{F}, \frac{\partial}{\partial y} \text{div}\vec{F}, \frac{\partial}{\partial z} \text{div}\vec{F}\right)$$

$$= (6x, 2, 1)$$

2C. $\vec{n} \cdot \nabla f|_{(x,y,z)} = (1,1,3)$

$$\vec{n} = \frac{1}{|\vec{v}|} \cdot \vec{v} = \frac{1}{\sqrt{4+1+4}} (2, 1, -2) = \frac{1}{3}(2, 1, -2)$$

$$\nabla f = (2xy^2z^2, \; 2yx^2z^2 + 4y, \; 2zx^2y^2)$$

$$\nabla f|_{1,1,3} = (18, 22, 18)$$

P. (2)

$\vec{n}\cdot\nabla f = \frac{1}{3}\begin{pmatrix}2\\1\\-2\end{pmatrix}\cdot\begin{pmatrix}18\\22\\18\end{pmatrix}$

$= \frac{22}{3}$

ii) Max rate of change

Greatest rate $= \sqrt{18^2+22^2+18^2}$

Direction of n must be in same direction of ∇f.

$= \sqrt{1132}$

3. $a\cdot\vec{n} = r\cdot\vec{n}$

$-z + y + 3 = 0$

$0x + y - z = -3$

$-3 = \begin{pmatrix}0\\1\\-1\end{pmatrix}\cdot\vec{n}$

$\vec{n} = \begin{pmatrix}0\\0\\3\end{pmatrix}$

Unit normal Vector $= \frac{1}{3}(0,0,3)$

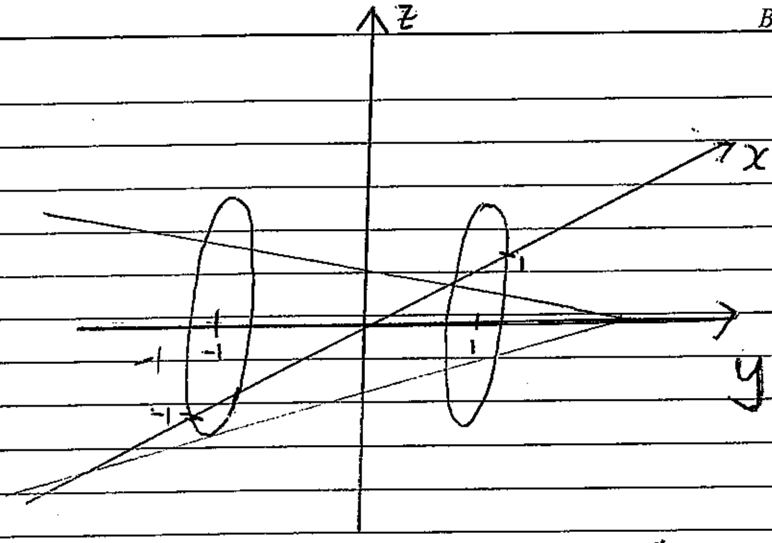
ii) $\iint_S g(x,y,z)\, d\sigma$

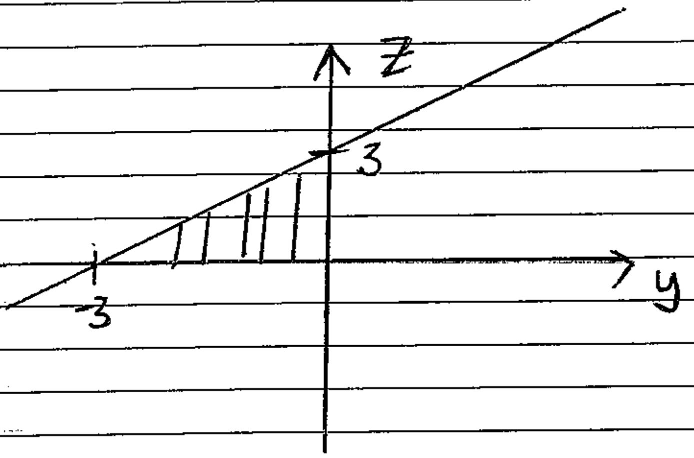
$$= \iint_R g(x,y,z(x,y)) \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dA$$

iii) Apologies, but please consult Prof.

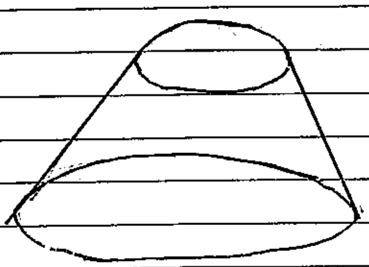
?P.(3)

3b.

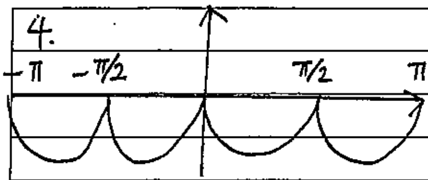




c i)



4.



$P = 2L = \pi/2$

$L = \pi/4$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{4}{\pi}\int_0^{\pi/2} \sin x \cdot \sin 4nx\, dx$$

$$= \frac{4}{\pi}\left(\frac{1}{2}\right)\int_0^{\pi/2} \cos(x - 4nx) - \cos(x + 4nx)\, dx$$

$$= \frac{2}{\pi}\left[\frac{\sin(x-4nx)}{1-4n} - \frac{\sin(x+4nx)}{1+4n}\right]_0^{\pi/2}$$

$$= \frac{2}{\pi}\left[\frac{(-1)^n}{1-4n} - \frac{(-1)^n}{1+4n}\right]$$

$$= \frac{2(-1)^n}{\pi}\left[\frac{1}{1-4n} - \frac{1}{1+4n}\right]$$

$$f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^n}{\pi}\left[\frac{1}{1-4n} - \frac{1}{1+4n}\right]\sin 4nx + \frac{2}{\pi}$$

c) at $x = \pi$, $f(\pi) = 0$

Look at the graph

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx$$

$$= \frac{4}{\pi}\int_0^{\pi/2} \sin x \cos 4nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi/2} \sin(x + 4nx) + \sin(x - 4nx)\, dx$$

$$= \frac{2}{\pi}\left[\frac{-\cos(x+4nx)}{1+4n} - \frac{\cos(x-4nx)}{1-4n}\right]_0^{\pi/2}$$

$$= 0$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi/2} \sin x\, dx$$

$$= \frac{2}{\pi}\left[-\cos x\right]_0^{\pi/2}$$

$$= \frac{2}{\pi}$$

7P.(4)

5a. $f(t) = e^{-2t}\sin t - e^{-2t}\sin(t)\,u(t-\pi)$

$= e^{-2t}\sin t - e^{-2t}\sin(t-\pi+\pi)\,u(t-\pi)$

$= e^{-2t}\sin t - e^{-2t}u(t-\pi)\left[\sin(t-\pi)\cos\pi + \cos(t-\pi)\sin\pi\right]$

$= e^{-2t}\sin t + e^{-2t}u(t-\pi)\sin(t-\pi)$

$$F(s) = \frac{1}{(s+2)^2+1} + e^{-\pi s}\left(\frac{1}{(s+2)^2+1}\right)$$

5b. $$L^{-1}\left[\frac{e^{-2s}}{s^2+4s+25}\right] = L^{-1}\left[\frac{e^{-2s}}{(s+2)^2-(\sqrt{21})^2}\right]$$

$$= \frac{e^{-2(t-2)}u(t-2)\sin(\sqrt{21}(t-2))}{\sqrt{21}}$$

5c. $y' - y = te^{-t}$, $y(0)=0$

$sY - y(0) - Y = L[te^{-t}]$

$sY - Y = \frac{1}{(s+1)^2}$

$Y(s-1) = \frac{1}{(s+1)^2}$

$$Y = \frac{1}{(s-1)(s+1)^2} = \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{(s+1)^2}$$

$1 = A(s+1)^2 + B(s-1)(s+1) + C(s-1)$

when $s=-1$, $C=-1/2$

$s=1$, $A=1/4$

$s=0$, $1 = 1/4 - B + 1/2$, $B=-1/4$

$$Y(s) = \frac{1}{4}\left(\frac{1}{s-1}\right) - \frac{1}{4}\left(\frac{1}{s+1}\right) - \frac{1}{2}\left(\frac{1}{(s+2)^2}\right)$$

$$y(t) = \frac{1}{4}L^{-1}\left[\frac{1}{s-1}\right] - \frac{1}{4}L^{-1}\left[\frac{1}{s+1}\right] - \frac{1}{2}L^{-1}\left[\frac{1}{(s+2)^2}\right]$$

$$= \frac{1}{4}e^{t} - \frac{1}{4}e^{-t} - \frac{1}{2}te^{-2t}$$

NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER 1 EXAMINATION 2017-2018

MA2006 – ENGINEERING MATHEMATICS

November/December 2017 Time Allowed: 2½ Hours

INSTRUCTIONS

1. This paper contains **FIVE (5)** questions and comprises **FIVE (5)** pages.
2. Answer **ALL FIVE (5)** questions.
3. Each questions carry equal marks.
4. This is a **CLOSED BOOK** examination.
5. Mathematical tables and formulae are provided on pages 4 and 5.

---

1 (a) Find real numbers x which make the matrix A below become a singular matrix.

$$A = \begin{bmatrix} 2 & 0 & 2 \\ 1 & 3x-4 & 2 \\ 0 & x^2-2 & 1 \end{bmatrix}$$

(5 marks)

(b) Find the eigenvalues and the associated eigenvectors of the matrix A.

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 25 & -1 & 28 & 0 \\ -10 & 0 & -8 & 0 \\ -37 & -8 & -29 & -5 \end{bmatrix}$$

(5 marks)

(c) Find the characteristic polynomial, the eigenvalues, and the associated eigenvectors of the matrix A. Find the matrix X that diagonalizes the matrix A ($D = X^{-1}AX$, *where D is diagonal matrix*) and diagonalize A. [Note: set the second components of the eigenvectors to be 1.]

$$A = \begin{bmatrix} -4 & 1 & -3 \\ 1 & -4 & -3 \\ -3 & -3 & 0 \end{bmatrix}$$

(10 marks)

ATTENTION: The Singapore Copyright Act applies to the use of this document. Nanyang Technological University Library

2 (a) Find the sum of the given pairs of vectors $\vec{x}$ and $\vec{y}$, their norms $\|\vec{x}\|$ and $\|\vec{y}\|$, their dot product $\vec{x}\cdot\vec{y}$, and their cross product $\vec{x}\times\vec{y}$.

$$\vec{x} = 2\vec{i} + 3\vec{j}\ ,\ \vec{y} = 3\vec{i} + \vec{j} - 2\vec{k}$$

(8 marks)

(b) Find $\phi = \operatorname{div}(\vec{v})$ and $\psi = \operatorname{curl}(\vec{v})$ for the vector function $\vec{v} = \cos(y)\vec{i} + (x^2+y^2)\vec{j} + e^{xz}\vec{k}$. Examine curl(grad(ϕ)) if the obtained function ϕ is smooth.

(12 marks)

3 (a) Find the line integral of the function $\vec{F} = (x^2+yz)\vec{i} + (y^2-xz)\vec{j} + (x^2+y^2)\vec{k}$ along the path given by $\Gamma: x = \sin(t), y = \cos(t), z = t$ from $t = 0$ to $t = \pi$.

(10 marks)

(b) Let S be the upper surface of the sphere $x^2 - 2x + y^2 + z^2 = 0$ and let Γ be the boundary. Find $\iint_S curl(\vec{F})\cdot d\vec{S}$ for $\vec{F} = (x^2 - y - 1)\vec{i} + xe^z\vec{j} + xy^2\vec{k}$.

(10 marks)

4. A periodic function is defined in one period as

$$f(x) = \begin{cases} -\sin\left(\frac{\pi x}{2}\right), & -2 \le x < -1; \\ x+1, & -1 \le x \le 0. \end{cases}$$

(a) Sketch $f(x)$ over the interval $-2 \le x \le 2$. (5 marks)

(b) Find the Fourier series for $f(x)$. (10 marks)

(c) Find the value of the Fourier series at $x = 1$. (5 marks)

24

5 (a) Find the Laplace transform of the function $f(t)$ defined as

$$f(t)=\begin{cases} t\cos(2t), & 0\le t<\pi; \\ e^{-t}, & t\ge\pi. \end{cases}$$

(5 marks)

(b) Find the inverse Laplace transform

$$L^{-1}\left\{\frac{e^{-3s}}{s^2-10s+24}\right\}.$$

(5 marks)

(c) Solve the following ordinary differential equation for $y(t)$ by using the Laplace transform.

$$y''-y=\begin{cases} e^{-2t}, & 0\le t<1; \\ 0, & t>1 \end{cases} \qquad y(0)=0,\ y'(0)=0.$$

(10 marks)

FORMULAE FOR FOURIER SERIES

Formulae for Fourier series for a periodic function $f(x)$ with period $P=2L$

$$f(x)=a_0+\sum_{n=1}^{\infty}(a_n\cos\frac{n\pi x}{L}+b_n\sin\frac{n\pi x}{L}), \qquad \text{where}$$

$$a_0=\frac{1}{2L}\int_{-L}^{L}f(x)dx,\ a_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos\frac{n\pi x}{L}dx,\ b_n=\frac{1}{L}\int_{-L}^{L}f(x)\sin\frac{n\pi x}{L}dx$$

Complex form of Fourier series for $f(x)$

$$f(x)=\sum_{n=-\infty}^{\infty}c_ne^{in\pi x/L} \quad \text{where} \quad c_n=\frac{1}{2L}\int_{-L}^{L}f(x)e^{-in\pi x/L}dx,\ n=0,\pm1,\pm2,\cdots$$

FORMULAE FOR SPECIAL FUNCTIONS

$\sin(x\pm y)=\sin x\cos y\pm\cos x\sin y$ $\qquad$ $\sin x\sin y=[-\cos(x+y)+\cos(x-y)]/2$

$\cos(x\pm y)=\cos x\cos y\mp\sin x\sin y$ $\qquad$ $\cos x\cos y=[\cos(x+y)+\cos(x-y)]/2$

$\sin x\cos y=[\sin(x+y)+\sin(x-y)]/2$

$\sin^2 x=(1-\cos 2x)/2,\quad \cos^2 x=(1+\cos 2x)/2$

$\sin x+\sin y=2\sin\frac{x+y}{2}\cos\frac{x-y}{2},\quad \sin x-\sin y=2\cos\frac{x+y}{2}\sin\frac{x-y}{2}$

$\cos x+\cos y=2\cos\frac{x+y}{2}\cos\frac{x-y}{2},\quad \cos x-\cos y=-2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$

$\sinh x=(e^x-e^{-x})/2,\quad \cosh x=(e^x+e^{-x})/2$

$\sinh x=-i\sin ix,\quad \cosh x=\cos ix$

LAPLACE TRANSFORM TABLE

$f(t)=L^{-1}\{F(s)\}$	$F(s)=L\{f(t)\}$	$f(t)=L^{-1}\{F(s)\}$	$F(s)=L\{f(t)\}$
1	$\frac{1}{s}$	$\sinh at$	$\frac{a}{s^2-a^2}$
t^{n-1}	$\frac{(n-1)!}{s^n},\ (n=1,2,\cdots)$	$\cosh at$	$\frac{s}{s^2-a^2}$
$t^{n-1}e^{at}$	$\frac{(n-1)!}{(s-a)^n},\ (n=1,2,\cdots)$	$u(t-a)$, Unit step function	$\frac{e^{-as}}{s}$
$\sin\omega t$	$\frac{\omega}{s^2+\omega^2}$	$\delta(t-a)$, Unit impulse function	e^{-as}
$\cos\omega t$	$\frac{s}{s^2+\omega^2}$		

LAPLACE TRANSFORM FORMULAE

$f(t)=L^{-1}\{F(s)\}$	$F(s)=L\{f(t)\}$	Remarks
$f^{(n)}(t)$	$s^n F(s)-s^{n-1}f(0)-\cdots-f^{(n-1)}(0)$	Differentiation of function
$\int_0^t f(\tau)d\tau$	$F(s)/s$	Integration of function
$e^{at}f(t)$	$F(s-a)$	Shift on s-axis
$u(t-a)f(t-a)$	$e^{-as}F(s)$	Shift on t-axis
$tf(t)$	$-F'(s)=-dF/ds$	Differentiation of transform
$t^n f(t)$	$(-1)^n F^{(n)}(s)$	Differentiation of transform
$f(t)/t$	$\int_s^\infty F(\tilde{s})d\tilde{s}$	Integration of transform
$\int_0^t f(t-\tau)g(\tau)d\tau$ $=\int_0^t f(\tau)g(t-\tau)d\tau$	$L(f*g)=L(f)L(g)$	Convolution

END OF PAPER

25

MA2006 - Engineering Mathematics

2017-18 Sem 1 (Nov/Dec 2017)

1/. a) Singular Matrix ⇒ Inverse does not exist ⇒ $\det(A) = 0$

$$\begin{vmatrix} 2 & 0 & 2 \\ 1 & 3x-4 & 2 \\ 0 & x^2-2 & 1 \end{vmatrix} = 0 \Rightarrow (2)(3x-4)(1) + (0)(2)(0) + (2)(1)(x^2-2) - (0)(3x-4)(2) - (x^2-2)(2)(2) - (1)(1)(0) = 0$$

$$6x - 8 + 2x^2 - 4 - 4x^2 + 8 = 0$$

$$2x^2 - 6x + 4 = 0$$

$$x^2 - 3x + 2 = 0$$

$$(x-1)(x-2) = 0 \Rightarrow x = 1 \text{ or } x = 2$$

b)

$$\begin{vmatrix} 2-\lambda & 0 & 0 & 0 \\ 25 & -1-\lambda & 28 & 0 \\ -10 & 0 & -8-\lambda & 0 \\ -37 & -8 & -29 & -5-\lambda \end{vmatrix} = 0 \Rightarrow (2-\lambda)(-1-\lambda)(-8-\lambda)(-5-\lambda) = 0$$

$$\lambda = 2 \text{ or } \lambda = -1 \text{ or } \lambda = -8 \text{ or } \lambda = -5$$

When $\lambda_1 = 2$,

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 25 & -3 & 28 & 0 \\ -10 & 0 & -10 & 0 \\ -37 & -8 & -29 & -7 \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{pmatrix} = 0$$

$a_1 = -t,\ b_1 = t,\ c_1 = t,\ d_1 = 0$

$\therefore X_1 = t\begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix},\ t \in \mathbb{R},\ t \neq 0$

When $\lambda_2 = -1$,

$$\begin{pmatrix} 3 & 0 & 0 & 0 \\ 25 & 0 & 28 & 0 \\ -10 & 0 & -7 & 0 \\ -37 & -8 & -29 & -4 \end{pmatrix}\begin{pmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{pmatrix} = 0$$

$a_2 = 0,\ b_2 = -t,\ c_2 = 0,\ d_2 = 2t$

$\therefore X_2 = t\begin{pmatrix} 0 \\ -1 \\ 0 \\ 2 \end{pmatrix},\ t \in \mathbb{R},\ t \neq 0$

When $\lambda_3 = -8$,

$$\begin{pmatrix} 10 & 0 & 0 & 0 \\ 25 & 7 & 28 & 0 \\ -10 & 0 & 0 & 0 \\ -37 & -8 & -29 & 3 \end{pmatrix}\begin{pmatrix} a_3 \\ b_3 \\ c_3 \\ d_3 \end{pmatrix} = 0$$

$a_3 = 0,\ b_3 = 4t,\ c_3 = -t,\ d_3 = t$

$\therefore X_3 = t\begin{pmatrix} 0 \\ 4 \\ -1 \\ 1 \end{pmatrix},\ t \in \mathbb{R},\ t \neq 0$

When $\lambda_4 = -5$,

$$\begin{pmatrix} 7 & 0 & 0 & 0 \\ 25 & 4 & 28 & 0 \\ -10 & 0 & -3 & 0 \\ -37 & -8 & -29 & 0 \end{pmatrix}\begin{pmatrix} a_4 \\ b_4 \\ c_4 \\ d_4 \end{pmatrix} = 0$$

$a_4 = 0,\ b_4 = 0,\ c_4 = 0,\ d_4 =$

$\therefore X_4 = t\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},\ t \in \mathbb{R},\ t \neq$

c)

$$\begin{vmatrix} -4-\lambda & 1 & -3 \\ 1 & -4-\lambda & -3 \\ -3 & -3 & -\lambda \end{vmatrix} = 0 \Rightarrow (-4-\lambda)(-4-\lambda)(-\lambda) + (1)(-3)(-3) + (-3)(1)(-3) - (-3)(-4-\lambda)(-3) - (-3)(-3)(-4-\lambda) - (-\lambda)(1)(1) = 0$$

$$-\lambda^3 - 8\lambda^2 - 16\lambda + 9 + 9 + 9\lambda + 36 + 9\lambda + 36 + \lambda = 0$$

$$-\lambda^3 - 8\lambda^2 + 3\lambda + 90 = 0$$

$$\lambda = 3 \text{ or } \lambda = -5 \text{ or } \lambda = -6$$

When $\lambda_1 = 3$,

$$\begin{pmatrix} -7 & 1 & -3 \\ 1 & -7 & -3 \\ -3 & -3 & -3 \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = 0$$

$a_1 = -t,\ b_1 = -t,\ c_1 = 2t$

$\therefore X_1 = t\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix},\ t \in \mathbb{R},\ t \neq 0$

When $\lambda_2 = -5$

$$\begin{pmatrix} 1 & 1 & -3 \\ 1 & 1 & -3 \\ -3 & -3 & 5 \end{pmatrix}\begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} = 0$$

$a_2 = -t,\ b_2 = t,\ c_2 = 0$

$\therefore X_2 = t\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix},\ t \in \mathbb{R},\ t \neq 0$

When $\lambda_3 = -6$

$$\begin{pmatrix} 2 & 1 & -3 \\ 1 & 2 & -3 \\ -3 & -3 & 6 \end{pmatrix}\begin{pmatrix} a_3 \\ b_3 \\ c_3 \end{pmatrix} = 0$$

$a_3 = t,\ b_3 = t,\ c_3 = t$

$\therefore X_3 = t\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},\ t \in \mathbb{R},\ t \neq 0$

Set $t = 1$, $X = \begin{pmatrix} -1 & -1 & 1 \\ -1 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -6 \end{pmatrix}$, $X^{-1} = \begin{pmatrix} -\frac{1}{6} & -\frac{1}{6} & \frac{1}{3} \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$ $\therefore A = XDX^{-1} = \begin{pmatrix} -1 & -1 & 1 \\ -1 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -6 \end{pmatrix}\begin{pmatrix} -\frac{1}{6} & -\frac{1}{6} & \frac{1}{3} \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$

2/. a) $\vec{x} + \vec{y} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ -2 \end{pmatrix} = 5\vec{i} + 4\vec{j} - 2\vec{k}$

$\vec{x} \times \vec{y} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} \times \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -6 \\ 4 \\ -7 \end{pmatrix} = -6\vec{i} + 4\vec{j} - 7\vec{k}$

$\|\vec{x}\| = \sqrt{2^2 + 3^2 + 0^2} = \sqrt{13}$ $\quad \|\vec{y}\| = \sqrt{3^2 + 1^2 + (-2)^2} = \sqrt{14}$

$\vec{x} \cdot \vec{y} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} = (2)(3) + (3)(1) + (0)(-2) = 9$

b)

$$\phi = \operatorname{div}(\vec{v}) = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \cdot \begin{pmatrix} \cos y \\ x^2+y^2 \\ e^{xz} \end{pmatrix} = 2y + xe^{xz}$$

$$\psi = \operatorname{curl}(\vec{v}) = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \times \begin{pmatrix} \cos y \\ x^2+y^2 \\ e^{xz} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial y} e^{xz} - \frac{\partial}{\partial z}(x^2+y^2) \\ \frac{\partial}{\partial z}\cos y - \frac{\partial}{\partial x} e^{xz} \\ \frac{\partial}{\partial x}(x^2+y^2) - \frac{\partial}{\partial y}\cos y \end{pmatrix} = \begin{pmatrix} 0 \\ -ze^{xz} \\ 2x + \sin y \end{pmatrix} = -ze^{xz}\vec{j} + (2x + \sin y)\vec{k}$$

$$\operatorname{grad}(\phi) = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix}(2y + xe^{xz}) = \begin{pmatrix} e^{xz} + xze^{xz} \\ 2 \\ x^2e^{xz} \end{pmatrix}$$

$$\operatorname{curl}(\operatorname{grad}(\phi)) = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \times \begin{pmatrix} e^{xz} + xze^{xz} \\ 2 \\ x^2e^{xz} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial y} x^2e^{xz} - \frac{\partial}{\partial z} 2 \\ \frac{\partial}{\partial z}(e^{xz} + xze^{xz}) - \frac{\partial}{\partial x} x^2e^{xz} \\ \frac{\partial}{\partial x} 2 - \frac{\partial}{\partial y}(e^{xz} + xze^{xz}) \end{pmatrix} = \begin{pmatrix} 0 \\ xe^{xz} + xe^{xz} + x^2ze^{xz} - 2xe^{xz} - x^2ze^{xz} \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

3/. a) $\Gamma: \ x = \sin t, \quad y = \cos t, \quad z = t, \quad t = 0 \text{ to } t = \pi$

$\frac{dx}{dt} = \cos t, \quad \frac{dy}{dt} = -\sin t, \quad \frac{dz}{dt} = 1$

$$\int_\Gamma \vec{F} \cdot d\vec{r} = \int_\Gamma \begin{pmatrix} x^2 + yz \\ y^2 - xz \\ x^2 + y^2 \end{pmatrix} \cdot \begin{pmatrix} \cos t \\ -\sin t \\ 1 \end{pmatrix} dt$$

$$= \int_0^\pi (\sin^2 t + t\cos t)(\cos t) + (\cos^2 t - t\sin t)(-\sin t) + (\sin^2 t + \cos^2 t) \ dt$$

$$= \int_0^\pi (\tfrac{1}{2} - \tfrac{1}{2}\cos 2t)(\cos t) + \tfrac{1}{2}t(1 + \cos 2t) + (\tfrac{1}{2} + \tfrac{1}{2}\cos 2t)(-\sin t) + \tfrac{1}{2}t(1 - \cos 2t) + 1 \ dt$$

$$= \int_0^\pi \tfrac{1}{2}\cos t - \tfrac{1}{2}\cos 2t\cos t + \tfrac{1}{2}t + \cancel{\tfrac{1}{2}t\cos 2t} - \tfrac{1}{2}\sin t - \tfrac{1}{2}\cos 2t \sin t + \tfrac{1}{2}t - \cancel{\tfrac{1}{2}t\cos 2t} + 1 \ dt$$

$$= \int_0^\pi (\tfrac{1}{2}\cos t - \tfrac{1}{2}\sin t + t + 1) - \tfrac{1}{4}\cos 3t - \tfrac{1}{4}\cos t - \tfrac{1}{4}\sin 3t + \tfrac{1}{4}\sin t \ dt$$

$$= \int_0^\pi \tfrac{1}{4}\cos t - \tfrac{1}{4}\sin t - \tfrac{1}{4}\cos 3t - \tfrac{1}{4}\sin 3t + t + 1 \ dt$$

$$= \left[\tfrac{1}{4}\sin t + \tfrac{1}{4}\cos t - \tfrac{1}{12}\sin 3t + \tfrac{1}{12}\cos 3t + \tfrac{1}{2}t^2 + t\right]_0^\pi$$

$$= \cancel{\tfrac{1}{4}\sin\pi} + \tfrac{1}{4}\cos\pi - \cancel{\tfrac{1}{12}\sin 3\pi} + \tfrac{1}{12}\cos 3\pi + \tfrac{1}{2}\pi^2 + \pi - \cancel{\tfrac{1}{4}\sin 0} - \tfrac{1}{4}\cos 0 + \cancel{\tfrac{1}{12}\sin 3(0)} - \tfrac{1}{12}\cos 3(0)$$

$$= -\tfrac{1}{4} - \tfrac{1}{12} + \tfrac{1}{2}\pi^2 + \pi - \tfrac{1}{4} - \tfrac{1}{12}$$

$$= \tfrac{1}{2}\pi^2 + \pi - \tfrac{2}{3} \quad //$$

b) $x^2 - 2x + y^2 + z^2 = 0$

$(x-1)^2 + y^2 + z^2 = 1 \implies$ sphere of radius 1 centred at $(1, 0, 0)$

$\therefore$ Boundary, Γ, is a unit circle on x-y plane centred at $(1, 0)$

$\Gamma: \ (x-1)^2 + y^2 = 1$

$x - 1 = \cos t, \quad y = \sin t, \quad z = 0$

$x = \cos t + 1 \qquad \frac{dy}{dt} = \cos t \qquad \frac{dz}{dt} = 0$

$\frac{dx}{dt} = -\sin t$

$$\iint_S \mathrm{curl}(\vec{F}) \cdot d\vec{S} = \iint_S \mathrm{curl}(\vec{F}) \cdot \hat{n} \, d\sigma$$

$$= \oint_\Gamma \vec{F} \cdot d\vec{r} \qquad \text{(by Stokes' Theorem)}$$

$$= \oint_\Gamma \begin{pmatrix} x^2 - y - 1 \\ xe^z \\ xy^2 \end{pmatrix} \cdot \begin{pmatrix} -\sin t \\ \cos t \\ 0 \end{pmatrix} dt$$

$$= \int_0^{2\pi} \left[(\cos t + 1)^2 - \sin t - 1\right](-\sin t) + (\cos t + 1)(\cos t) \ dt$$

$$= \int_0^{2\pi} (\cos^2 t + 2\cos t + 1 - \sin t - 1)(-\sin t) + \cos^2 t + \cos t \ dt$$

$$= \int_0^{2\pi} -\tfrac{1}{2}\sin t - \tfrac{1}{2}\cos 2t \sin t - 2\cos t \sin t + \sin^2 t + \cos^2 t + \cos t \ dt$$

$$= \int_0^{2\pi} -\tfrac{1}{2}\sin t - \tfrac{1}{4}\sin 3t + \tfrac{1}{4}\sin t - \sin 2t + 1 + \cos t \ dt$$

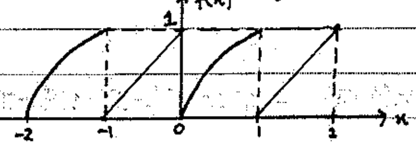
$$= \left[\tfrac{1}{2}\cos t + \tfrac{1}{12}\cos 3t - \tfrac{1}{4}\cos t + \tfrac{1}{2}\cos 2t + t + \sin t\right]_0^{2\pi}$$

$$= \tfrac{1}{2}\cos 2\pi + \tfrac{1}{12}\cos 6\pi - \tfrac{1}{4}\cos 2\pi + \tfrac{1}{2}\cos 4\pi + 2\pi + \sin 2\pi - \tfrac{1}{2} - \tfrac{1}{12} + \tfrac{1}{4} - \tfrac{1}{2}$$

$$= 2\pi \quad //$$

$$f(x) = \begin{cases} -\sin\left(\frac{\pi x}{2}\right), & -2\le x<-1 \\ x+1, & -1\le x\le 0 \end{cases}$$

4. a)



$P = 2L = 2$

$L = 1$

b) $a_0 = \frac{1}{2}\int_{-1}^{1} f(x)\,dx = \frac{1}{2}\int_{-1}^{0} x+1\,dx + \frac{1}{2}\int_0^1 \sin\frac{\pi x}{2}\,dx$

$= \frac{1}{2}\left[\frac{1}{2}x^2+x\right]_{-1}^{0} + \frac{1}{2}\left[\frac{-2}{\pi}\cos\frac{\pi x}{2}\right]_0^1$

$= \frac{1}{4} + \frac{1}{\pi}$

$a_n = \frac{1}{1}\int_{-1}^{1} f(x)\cos\frac{n\pi x}{1}\,dx$

$= \int_{-1}^{0}(x+1)\cos n\pi x\,dx + \int_0^1 \sin\frac{\pi x}{2}\cos n\pi x\,dx$

$= \int_{-1}^{0} x\cos n\pi x\,dx + \int_{-1}^{0}\cos n\pi x\,dx + \frac{1}{2}\int_0^1 \sin(n+\frac{1}{2})\pi x + \sin(-n+\frac{1}{2})\pi x\,dx$

$= \left[x\frac{1}{n\pi}\sin n\pi x\right]_{-1}^{0} - \int_{-1}^{0}\frac{1}{n\pi}\sin n\pi x\,dx + \left[\frac{1}{n\pi}\sin n\pi x\right]_{-1}^{0} - \frac{1}{2}\left[\frac{\cos(n+\frac{1}{2})\pi x}{(n+\frac{1}{2})\pi} + \frac{\cos(-n+\frac{1}{2})\pi x}{(-n+\frac{1}{2})\pi}\right]_0^1$

$= \left[\frac{1}{(n\pi)^2}\cos n\pi x\right]_{-1}^{0} - \frac{1}{2}\left[\frac{\cos(n+\frac{1}{2})\pi}{(n+\frac{1}{2})\pi} + \frac{\cos(-n+\frac{1}{2})\pi}{(-n+\frac{1}{2})\pi} - \frac{1}{(n+\frac{1}{2})\pi} - \frac{1}{(-n+\frac{1}{2})\pi}\right]$

$= \frac{1}{(n\pi)^2} - \frac{(-1)^n}{(n\pi)^2} + \frac{1}{\pi(2n+1)} - \frac{1}{\pi(2n-1)}$

$= \frac{1}{(n\pi)^2}\left[1-(-1)^n\right] - \frac{2}{\pi(4n^2-1)}$

$b_n = \frac{1}{1}\int_{-1}^{1} f(x)\sin\frac{n\pi x}{1}\,dx$

$= \int_{-1}^{0}(x+1)\sin n\pi x + \int_0^1 \sin\frac{\pi x}{2}\sin n\pi x\,dx$

$= \int_{-1}^{0} x\sin n\pi x\,dx + \int_{-1}^{0}\sin n\pi x\,dx + \frac{1}{2}\int_0^1 -\cos(n+\frac{1}{2})\pi x + \cos(n-\frac{1}{2})\pi x\,dx$

$= \left[x\frac{-1}{n\pi}\cos n\pi x\right]_{-1}^{0} - \int_{-1}^{0}\frac{-1}{n\pi}\cos n\pi x\,dx + \frac{-1}{n\pi}\left[\cos n\pi x\right]_{-1}^{0} + \frac{1}{2}\left[\frac{-\sin(n+\frac{1}{2})\pi x}{(n+\frac{1}{2})\pi} + \frac{\sin(n-\frac{1}{2})\pi x}{(n-\frac{1}{2})\pi}\right]_0^1$

$= -\frac{(-1)^n}{n\pi} + \frac{1}{(n\pi)^2}\left[\sin n\pi x\right]_{-1}^{0} + \frac{-1}{n\pi} + \frac{(-1)^n}{n\pi} - \frac{(-1)^n}{\pi(2n+1)} - \frac{(-1)^n}{\pi(2n-1)}$

$= \frac{-1}{n\pi} - \frac{(-1)^n}{\pi}\left[\frac{4n}{4n^2-1}\right]$

$\therefore f(x) = \frac{1}{4} + \frac{1}{\pi} + \sum_{n=1}^{\infty}\left\{\left[\frac{1}{(n\pi)^2}\left[1-(-1)^n\right] - \frac{2}{\pi(4n^2-1)}\right]\cos n\pi x + \left[\frac{-1}{n\pi} - \frac{(-1)^n}{\pi}\left(\frac{4n}{4n^2-1}\right)\right]\sin n\pi x\right\}$ //

c) From graph, when $x=1$; $f(x) = \frac{1}{2}(1+0) = \frac{1}{2}$ //

CASIO

5. a) $f(t) = t\cos 2t + (e^{-t} - t\cos 2t)\, u(t-\pi)$

$= t\cos 2t + \left[\frac{e^{-(t-\pi)}}{e^{\pi}} - (t-\pi)\cos 2(t-\pi)\right] u(t-\pi) - \pi\cos 2(t-\pi)\, u(t-\pi)$

$L(t\cos 2t) = \frac{-d}{ds} L(\cos 2t) = \frac{-d}{ds}\frac{s}{s^2+4} = \frac{-(s^2+4)(1)+(s)(2s)}{(s^2+4)^2} = \frac{s^2-4}{(s^2+4)^2}$

$L[e^{-(t-\pi)} u(t-\pi)] = e^{-\pi s}\frac{1}{s+1}$

$L[(t-\pi)\cos 2(t-\pi)\, u(t-\pi)] = e^{-\pi s} L[t\cos 2t] = e^{-\pi s}\frac{s^2-4}{(s^2+4)^2}$

$L[\cos 2(t-\pi)\, u(t-\pi)] = e^{-\pi s} L[\cos 2t] = e^{-\pi s}\frac{s}{s^2+4}$

$\therefore F(s) = \frac{s^2-4}{(s^2+4)^2} + \frac{e^{-\pi(s+1)}}{s+1} - \frac{e^{-\pi s}(s^2-4)}{(s^2+4)^2} - \frac{e^{-\pi s}\,\pi s}{s^2+4}$ //

b) Let $f(t) = L^{-1}\left\{\frac{e^{-3s}}{s^2-10s+24}\right\} = L^{-1}[F(s)]$

$F(s) = \frac{e^{-3s}}{s^2-10s+24} = e^{-3s}\frac{1}{(s-6)(s-4)} = e^{-3s}\left[\frac{\frac{1}{2}}{s-6} + \frac{-\frac{1}{2}}{s-4}\right]$

$\therefore f(t) = \left[\frac{1}{2}e^{6(t-3)} - \frac{1}{2}e^{4(t-3)}\right] u(t-3)$ //

c) $y'' - y = e^{-2t}$, $0 \le t < 1$ $y(0)=0$, $y'(0)=0$

$L(y'') - L(y) = L(e^{-2t})$

$s^2L(y) - s\,y(0) - y'(0) - L(y) = \frac{1}{s+2}$ (with $s\,y(0)$ and $y'(0)$ crossed out)

$(s^2-1)L(y) = \frac{1}{s+2}$

$L(y) = \frac{1}{(s^2-1)(s+2)} = \frac{A}{(s+1)} + \frac{B}{(s-1)} + \frac{C}{(s+2)}$

$= \frac{As^2 + As - 2A + Bs^2 + 3Bs + 2B + Cs^2 - C}{(s^2-1)(s+2)}$

By comparing,
s^2: $A + B + C = 0$
s: $A + 3B = 0$
Constant: $-2A + 2B - C = 1$

$A = \frac{-1}{2}$, $B = \frac{1}{6}$, $C = \frac{1}{3}$

$\therefore L(y) = \frac{-\frac{1}{2}}{s+1} + \frac{\frac{1}{6}}{s-1} + \frac{\frac{1}{3}}{s+2}$

$y = \frac{-1}{2}e^{-t} + \frac{1}{6}e^{t} + \frac{1}{3}e^{-2t}$

$y'' - y = 0$, $t > 1$

$L(y) = 0$

$y = 0$

$\therefore y = \frac{-1}{2}e^{-t} + \frac{1}{6}e^{t} + \frac{1}{3}e^{-2t} - \left[\frac{-1}{2}e^{-t} + \frac{1}{6}e^{t} + \frac{1}{3}e^{-2t}\right]u(t-1)$ //

NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER 2 EXAMINATION 2017-2018

MA2006 – ENGINEERING MATHEMATICS

April/May 2018 Time Allowed: 2½ hours

INSTRUCTIONS

1. This paper contains **FIVE (5)** questions and comprises **SIX (6)** pages.
2. Answer **ALL** questions.
3. All questions carry equal marks.
4. This is a **CLOSED-BOOK** examination.
5. Mathematical tables and formulae are provided on pages 4, 5 and 6.

1(a) Two 3×3 matrices A, B satisfy the relation: $A^{-1}BA = 6A + BA$. Given:

$$A = \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1/7 \end{bmatrix}.$$

Find matrix B. **Hint**: $A^{-1}BA - BA = (A^{-1} - I)BA$

(10 marks)

(b) It is known that the below linear system has infinitely many solutions:

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -a \\ b \\ -c \\ d \end{bmatrix}$$

Perform legitimate row operations to find the relationship between a, b, c, d.

(10 marks)

28

2(a) With reference to a Cartesian coordinate system $Oxyz$, the displacement of a point mass from the origin O is given by $\underline{\mathbf{r}}(t) = At^2\underline{\mathbf{i}} + Bt^3\underline{\mathbf{j}} + Ct^3\underline{\mathbf{k}}$ (in metre), where t is time (in second), A, B and C are constants and $\underline{\mathbf{i}}$, $\underline{\mathbf{j}}$ and $\underline{\mathbf{k}}$ are the Cartesian base vectors. If the point mass represents a one kilogram body and if it is acted upon by a force given by $\underline{\mathbf{i}} + 2\underline{\mathbf{j}} - 5\underline{\mathbf{k}}$ (in newton) at $t = 1$, calculate the velocity of the point mass at $t = 2$.

(10 marks)

(b) A positive charge in an electric field tends to move in the direction along which there is a maximum drop in electric potential. If the electric potential is given by $V(x, y, z) = x^2 - y^2 + xy + z$, which direction will the positive charge move if it is at the point (2,1,3)? (Give a vector $a\underline{\mathbf{i}} + b\underline{\mathbf{j}} + c\underline{\mathbf{k}}$ which points in the required direction).

(10 marks)

3. In all the parts below, Cartesian coordinates are denoted by the usual symbols x, y and z.

(a) Points (x, y, z) on the curve C are given by $x = 4 + t^2$, $y = t^2 + t + 1$ and $z = t$ for $1 \le t \le 2$. Evaluate the following line integrals over C.

$$\int_C (16z - z^2 + x)\, ds .$$

(10 marks)

(b) Construct a function $\phi(x, y)$ such that $\text{grad}(\phi) = (6x^2 + 6y)\underline{\mathbf{i}} + (6x + 6y)\underline{\mathbf{j}}$. Hence, evaluate the line integral $\int_C ((6x^2 + 6y)dx + (6x + 6y)dy)$, where C is a curve on the Oxy plane whose starting and ending points are $(-1,2)$ and $(-2,1)$ respectively.

(10 marks)

4. A periodic function is defined in one period as

$$f(x)=\begin{cases}\frac{1}{2}+2x, & 0\le x<\frac{1}{2};\\ 1-x, & \frac{1}{2}\le x\le 1.\end{cases}$$

(a) Sketch $f(x)$ over the interval $-1\le x\le 1$.

(5 marks)

(b) Find the Fourier series for $f(x)$.

(10 marks)

(c) Find the values of the Fourier series at $x = 1/2$ and $x = 2$.

(5 marks)

5(a) Find the Laplace transform of the function $f(t)$ defined as

$$f(t)=\begin{cases}te^{-2t}, & 0\le t<1;\\ 0, & t\ge 1.\end{cases}$$

(5 marks)

(b) Find the inverse Laplace transform

$$L^{-1}\left\{\frac{e^{-s}}{s^2-8s+17}\right\}.$$

(5 marks)

(c) Solve the following ordinary differential equation for $y(t)$ by using the Laplace transform.

$$y'-y=\begin{cases}0, & 0\le t<2\pi;\\ \sin t, & 2\pi<t\le 3\pi\\ 0, & t>3\pi\end{cases}\qquad y(0)=0.$$

(10 marks)

29

FORMULAE FOR VECTOR CALCULUS PART

1. $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}, \quad \vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k},$

$$\vec{a} \bullet \vec{b} = a_1b_1 + a_2b_2 + a_3b_3,$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - b_2a_3)\vec{i} + (a_3b_1 - b_3a_1)\vec{j} + (a_1b_2 - b_1a_2)\vec{k}.$$

2. $grad\ f = \nabla f = \dfrac{\partial f}{\partial x}\vec{i} + \dfrac{\partial f}{\partial y}\vec{j} + \dfrac{\partial f}{\partial z}\vec{k}$, for scalar function $f(x,y,z)$.

3. $\vec{V} = p(x,y,z)\vec{i} + q(x,y,z)\vec{j} + r(x,y,z)\vec{k},$

$$\nabla \bullet \vec{V} = div\,\vec{V} = \frac{\partial p}{\partial x} + \frac{\partial q}{\partial y} + \frac{\partial r}{\partial z},$$

$$\nabla \times \vec{V} = curl\,\vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ p & q & r \end{vmatrix} = \left(\frac{\partial r}{\partial y} - \frac{\partial q}{\partial z}\right)\vec{i} + \left(\frac{\partial p}{\partial z} - \frac{\partial r}{\partial x}\right)\vec{j} + \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right)\vec{k},$$

4. $ds = \sqrt{(dx)^2 + (dy)^2 + (dz)^2} = \sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2 + \left(\dfrac{dz}{dt}\right)^2}\,dt,$

$$\int_C f(x,y,z)ds = \int_{t_1}^{t_2} f[x(t),y(t),z(t)]\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt.$$

5. $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}, \quad d\vec{r} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k},$

For $\vec{F} = p(x,y,z)\vec{i} + q(x,y,z)\vec{j} + r(x,y,z)\vec{k},$

$$\int_C \vec{F} \bullet d\vec{r} = \int p\,dx + q\,dy + r\,dz = \int_{t_1}^{t_2}\left(p\frac{dx}{dt} + q\frac{dy}{dt} + r\frac{dz}{dt}\right)dt.$$

6. Green's Theorem

$$\oint_C f(x,y)dx + g(x,y)dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)dA$$

7. surface integral:

$$\iint_S g(x,y,z)d\sigma = \iint_R g[x,y,z(x,y)]\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\,dA,$$

for the surface given by $z = f(x,y)$.

FORMULAE FOR SPECIAL FUNCTIONS

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y \qquad \sin x \sin y = [-\cos(x+y) + \cos(x-y)]/2$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y \qquad \cos x \cos y = [\cos(x+y) + \cos(x-y)]/2$$

$$\sin x \cos y = [\sin(x+y) + \sin(x-y)]/2$$

$$\sin^2 x = (1 - \cos 2x)/2, \qquad \cos^2 x = (1 + \cos 2x)/2$$

$$\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}, \qquad \sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}$$

$$\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}, \qquad \cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$$

$$\sinh x = (e^x - e^{-x})/2, \qquad \cosh x = (e^x + e^{-x})/2$$

$$\sinh x = -i \sin ix, \qquad \cosh x = \cos ix$$

FORMULAE FOR FOURIER SERIES AND TRANSFORM

Euler Formulae for Fourier series for a periodic function $f(x)$ with a period $P = 2L$

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

where $a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx$, $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx$, $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx$

Complex form of Fourier series for $f(x)$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L} \quad \text{where} \quad c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}dx, \; n = 0, \pm 1, \pm 2, \cdots$$

Fourier integral

$$f(x) = \int_0^{\infty}[A(\omega)\cos\omega x + B(\omega)\sin\omega x]d\omega$$

where $A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos\omega v dv$ and $B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin\omega v dv$

Complex form of Fourier integral

$$f(x) = \int_{-\infty}^{\infty} C(\omega)e^{i\omega x}d\omega, \qquad C(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx$$

Cosine transform

$$\hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos(\omega x)dx, \qquad f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} \hat{f}_c(\omega)\cos(\omega x)d\omega$$

Sine transform

$$\hat{f}_s(\omega) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin(\omega x)dx, \qquad f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} \hat{f}_s(\omega)\sin(\omega x)d\omega$$

Fourier transform

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx, \qquad f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega x}d\omega$$

LAPLACE TRANSFORM TABLE

$f(t)=L^{-1}\{F(s)\}$	$F(s)=L\{f(t)\}$	$f(t)=L^{-1}\{F(s)\}$	$F(s)=L\{f(t)\}$
1	$\frac{1}{s}$	$\frac{\sinh at}{a}$	$\frac{1}{s^2-a^2}$
$\frac{t^{n-1}}{(n-1)!}$	$\frac{1}{s^n},\ (n=1,2,\mathrm{L}\ )$	$\cosh at$	$\frac{s}{s^2-a^2}$
$\frac{t^{n-1}e^{at}}{(n-1)!}$	$\frac{1}{(s-a)^n},\ (n=1,2,\mathrm{L}\ )$	$u(t-a)$, Unit step function	$\frac{e^{-as}}{s}$
$\frac{\sin \omega t}{\omega}$	$\frac{1}{s^2+\omega^2}$	$\delta(t-a)$, Unit impulse function	e^{-as}
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$		

LAPLACE TRANSFORM FORMULAE

$f(t)=L^{-1}\{F(s)\}$	$F(s)=L\{f(t)\}$	Remarks
$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-\mathrm{L}\ -f^{(n-1)}(0)$	Differentiation of function
$\int_0^t f(\tau)d\tau$	$F(s)/s$	Integration of function
$e^{at}f(t)$	$F(s-a)$	Shift on s-axis
$u(t-a)f(t-a)$	$e^{-as}F(s)$	Shift on t-axis
$tf(t)$	$-F'(s)=-dF/ds$	Differentiation of transform
$t^nf(t)$	$(-1)^nF^{(n)}(s)$	Differentiation of transform
$f(t)/t$	$\int_S^{\infty} F(S)ds$	Integration of transform
$\int_0^t f(t-\tau)g(\tau)d\tau$ $=\int_0^t f(\tau)g(t-\tau)d\tau$	$L(f*g)=L(f)L(g)$	Convolution
$f(t)=f(t+p)$	$\int_0^p e^{-st}f(t)dt\Big/(1-e^{-sp})$	$f(t)$ is periodic with period p

END OF PAPER

Q1)(a) $A^{-1}BA = 6A + BA$ — ①

$$A = \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1/7 \end{bmatrix}$$

From ①:

$A(A^{-1}BA)A^{-1} = A(6A + BA)A^{-1}$

$IBI = 6AAA^{-1} + ABAA^{-1}$

$B = 6AI + ABI$ $(AA^{-1} = I)$

$B = 6A + AB$

$B - AB = 6A$

$B(I - A) = 6A$ — ②

$$\therefore I - A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1/7 \end{pmatrix} = \begin{pmatrix} 2/3 & 0 & 0 \\ 0 & 3/4 & 0 \\ 0 & 0 & 6/7 \end{pmatrix}$$

$$\therefore (I - A)^{-1} = \begin{pmatrix} 2/3 & 0 & 0 \\ 0 & 3/4 & 0 \\ 0 & 0 & 6/7 \end{pmatrix}^{-1} = \begin{pmatrix} 3/2 & 0 & 0 \\ 0 & 4/3 & 0 \\ 0 & 0 & 7/6 \end{pmatrix}$$

Hence, from ②, $B = 6A(I - A)^{-1}$

$$= 6\begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1/7 \end{pmatrix}\begin{pmatrix} 3/2 & 0 & 0 \\ 0 & 4/3 & 0 \\ 0 & 0 & 7/6 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3/2 & 0 \\ 0 & 0 & 6/7 \end{pmatrix}\begin{pmatrix} 3/2 & 0 & 0 \\ 0 & 4/3 & 0 \\ 0 & 0 & 7/6 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(b)
$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -a \\ b \\ -c \\ d \end{bmatrix}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & -a \\ 0 & 1 & 1 & 0 & b \\ 0 & 0 & 1 & 1 & -c \\ 1 & 0 & 0 & 1 & d \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & -a \\ 0 & 1 & 1 & 0 & b \\ 0 & 0 & 1 & 1 & -c \\ 0 & -1 & 0 & 1 & d+a \end{array}\right) R_4 \to R_4 - R_1 \longrightarrow \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & -a \\ 0 & 1 & 1 & 0 & b \\ 0 & 0 & 1 & 1 & -c \\ 0 & 0 & 1 & 1 & d+a+b \end{array}\right) R_4 \to R_4 + R_2$$

$$\longrightarrow \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & -a \\ 0 & 1 & 1 & 0 & b \\ 0 & 0 & 1 & 1 & -c \\ 0 & 0 & 0 & 0 & d+a+b+c \end{array}\right) R_4 \to R_4 - R_3$$

Hence, as linear system has indefinitely many solution, $a + b + c + d = 0$

DISCLAIMER: *The solutions are done by students who scored A or above in this subject. MAE Club and Campus supplies are not liable or responsible for any errors in the contents of these solutions. Students are advised to take the solutions as a guide rather than absolute answers to exam paper.*

Should there be any mistake identified, please proceed to the Facebook link encoded in the QR code to feedback or submit correct answers. The link is: http://bit.ly/2lW2C32

Q2) (a) $\underline{r}(t) = At^2\underline{i} + Bt^3\underline{j} + Ct^3\underline{k}$

let d be displacement, v be velocity, a be acceleration.

Hence, $d_x = At^2$ → $V_x = 2At$ → $a_x = 2A$

$d_y = Bt^3$ → (differentiate w.r.t. time, t.) $V_y = 3Bt^2$ → $a_y = 6B$

$d_z = Ct^3$ → $V_z = 3Ct^2$ → $a_z = 6C$

For 1 kg mass, $F = \underline{i} + 2\underline{j} - 5\underline{k}$

∴ By comparison, $2A = 1$, $2 = 6B$, $-5 = 6C$

$A = \frac{1}{2}$, $B = \frac{1}{3}$, $C = -\frac{5}{6}$

∴ $V_x = 2(\frac{1}{2})t$ at $t=2$, $V_x = 2(\frac{1}{2})(2) = 2$

$V_y = 3(\frac{1}{3})t^2$, $V_y = 3(\frac{1}{3})(2)^2 = 4$

$V_z = 3(-\frac{5}{6})t^2$, $V_z = 3(-\frac{5}{6})(2)^2 = -10$

Hence, at $t=2$, velocity, $v(t) = \underline{2\underline{i} + 4\underline{j} - 10\underline{k}}$

(b) $V(x,y,z) = x^2 - y^2 + xy + z$

$\nabla V(x,y,z) = [2x + y, -2y + x, 1]$

For maximum increase in potential, we move in direction of gradient vector at [2,1,3]. Hence, for maximum drop, we move in opposite direction.

∴ direction for maximum drop $= -\nabla V(2,1,3)$

$= -[2(2)+1, -2(1)+2, 1]$

$= -[5, 0, 1]$

$= [-5, 0, -1]$

∴ direction is $\underline{-5\underline{i} - \underline{k}}$

Q3) (a) $\int_c (16z - z^2 + x)\, ds$ — ① $\quad x = 4 + t^2 \quad \frac{dx}{dt} = 2t$

for $1 \le t \le 2$ $\quad y = t^2 + t + 1 \Rightarrow \frac{dy}{dt} = 2t + 1$

$z = t \quad \frac{dz}{dt} = 1$

Hence, from ①,

$\int_c (16z - z^2 + x)\, ds \qquad ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$

$= \int_1^2 16t - t^2 + (4 + t^2)\sqrt{8t^2 + 4t + 2}\, dt \qquad = \sqrt{(2t)^2 + (2t+1)^2 + 1^2}\, dt$

$= \sqrt{4t^2 + 4t^2 + 4t + 1 + 1}\, dt$

$= \int_1^2 16t + 4\sqrt{8t^2 + 4t + 2}\, dt \qquad = \sqrt{8t^2 + 4t + 2}\, dt$

$= \left[(8t^2 + 4t + 2)^{3/2} \left(\frac{2}{3}\right) \right]_1^2$

$= \frac{2}{3}\left[42^{3/2} - 14^{3/2} \right]$

$= \frac{2}{3}\left[42\sqrt{42} - 14\sqrt{14} \right]$

$= 28\sqrt{42} - \frac{28}{3}\sqrt{14}$

$= 28\left[\sqrt{42} - \frac{1}{3}\sqrt{14} \right]$

(b) $\text{grad}(\phi) = (6x^2 + 6y)\,\underline{i} + (6x + 6y)\,\underline{j}$

Let $\frac{dM}{dx} = 6x^2 + 6y$ — ① and $\frac{dN}{dy} = 6x + 6y$ — ②

Hence from $\frac{dM}{dx} = 6x^2 + 6y$ $\qquad$ comparing ② and ③,

$M = \int 6x^2 + 6y\, dx \qquad f'(y) = 6y$

$= 2x^3 + 6xy + f(y) \qquad f(y) = 3y^2 + c$

differentiate w.r.t y, $\frac{dM}{dy} = 6x + f'(y)$ — ③

Hence, $\phi = 2x^3 + 6xy + 3y^2 + c$

from $\int_c (6x^2 + 6y)\,dx + (6x + 6y)\,dy$)

$g(x,y) = 6x + 6y \qquad f(x,y) = 6x^2 + 6y$

$\frac{\partial g}{\partial x} = 6 \qquad \frac{\partial f}{\partial x} = 6$

as $\frac{\partial g}{\partial x} = \frac{\partial f}{\partial y}$, we can say that line integral is path-independent.

Hence, $\int_c (6x^2 + 6y)\,dx + (6x + 6y)\,dy = \phi(-2,1) - \phi(-1,2)$

$= \left(2(-2)^3 + 6(-2)(1) + 3(1)^2 + c\right) - \left(2(-1)^3 + 6(-1)(2) + 3(2)^2 + c\right)$

$= (-25 + c) - (-2 + c)$

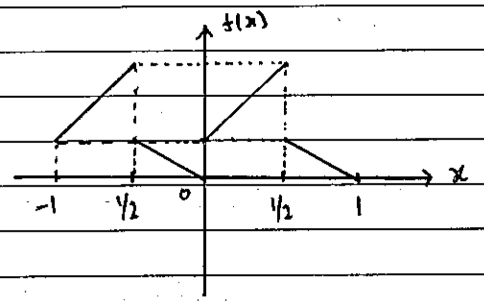
$= -25 + c + 2 - c$

$= -23$

DISCLAIMER: *The solutions are done by students who scored A or above in this subject. MAE Club and Campus supplies are not liable or responsible for any errors in the contents of these solutions. Students are advised to take the solutions as a guide rather than absolute answers to exam paper.*

Should there be any mistake identified, please proceed to the Facebook link encoded in the QR code to feedback or submit correct answers. The link is: http://bit.ly/2lW2C32

Q4) (a)



$$f(x) = \begin{cases} \frac{1}{2} + 2x, & 0 \le x < 1/2 \\ 1 - x, & 1/2 \le x \le 1 \end{cases}$$

(b) $f(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right) \right]$

$p = 2L = 1 \Rightarrow L = 1/2$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2(\frac{1}{2})}\int_0^{1/2} \frac{1}{2} + 2x\,dx + \frac{1}{2(\frac{1}{2})}\int_{1/2}^{1} 1 - x\,dx$$

$$= \left[\tfrac{1}{2}x + x^2\right]_0^{1/2} + \left[x - \tfrac{1}{2}x^2\right]_{1/2}^{1}$$

$$= 0.5 - 0 + 0.5 - 0.375$$

$$= 0.625 = \frac{5}{8}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx$$

$$= \frac{1}{1/2}\int_0^{1/2} \left(\tfrac{1}{2} + 2x\right)\cos\left(\frac{n\pi}{1/2}x\right)dx + \frac{1}{1/2}\int_{1/2}^{1} (1-x)\cos\left(\frac{n\pi}{1/2}x\right)dx$$

$$= 2\int_0^{1/2} \left(\tfrac{1}{2} + 2x\right)\cos(2n\pi x)\,dx + 2\int_{1/2}^{1} (1-x)\cos(2n\pi x)\,dx$$

$$= 2\left[\left(\tfrac{1}{2}+2x\right)\left(\tfrac{1}{2n\pi}\right)(\sin(2n\pi x))\right]_0^{1/2} - 2\int_0^{1/2} 2\left(\tfrac{1}{2n\pi}\right)\sin(2n\pi x)\,dx$$

$$+ 2\left[(1-x)\left(\tfrac{1}{2n\pi}\right)\sin(2n\pi x)\right]_{1/2}^{1} - 2\int_{1/2}^{1} (-1)\left(\tfrac{1}{2n\pi}\right)\sin(2n\pi x)\,dx$$

$$= 0 - 2\left[\frac{-1}{2n^2\pi^2}\cos(2n\pi x)\right]_0^{1/2} + 0 + \left[\frac{-1}{2n^2\pi^2}\cos(2n\pi x)\right]_{1/2}^{1}$$

$$= \left[\frac{1}{n^2\pi^2}\right]\left[\cos(n\pi) - 1\right] - \left(\frac{1}{2n^2\pi^2}\right)\left(\cos(2n\pi) - \cos(n\pi)\right]$$

$$= \left(\frac{1}{n^2\pi^2}\right)\left((-1)^n - 1\right] - \left(\frac{1}{2n^2\pi^2}\right)\left[1 - (-1)^n\right]$$

$$= \frac{1}{2n^2\pi^2}\left(3(-1)^n - 3\right)$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

$$= 2\int_0^{1/2} \left(\tfrac{1}{2}+2x\right)\sin(2n\pi x)\,dx + 2\int_{1/2}^{1} (1-x)\sin(2n\pi x)\,dx$$

$$= 2\left[\left(\tfrac{1}{2}+2x\right)\left(\frac{-1}{2n\pi}\right)\cos(2n\pi x)\right]_0^{1/2} - 2\int_0^{1/2} 2\left(\frac{-1}{2n\pi}\right)\cos(2n\pi x)\,dx$$

$$+ 2\left[(1-x)\frac{-1}{2n\pi}\cos(2n\pi x)\right]_{1/2}^{1} - 2\int_{1/2}^{1} (-1)\left(\frac{-1}{2n\pi}\right)\cos(2n\pi x)\,dx$$

$$= \frac{-3}{2n\pi}\cos(n\pi) + \frac{1}{2n\pi} + 2\left[\frac{1}{2n^2\pi^2}\sin(2n\pi x)\right]_0^{1/2} + 0 + \frac{1}{2n\pi}\cos(n\pi) - \left[\frac{1}{2n^2\pi^2}\sin(2n\pi x)\right]_{1/2}^{1}$$

$$= \frac{1}{2n\pi}\left(1 - 3(-1)^n\right) + 0 + \frac{1}{2n\pi}(-1)^n - 0$$

$$= \frac{3}{2n\pi}$$

$$\therefore f(x) = \frac{5}{8} + \sum_{n=1}^{\infty}\left[\frac{3((-1)^n - 1)}{2n^2\pi^2}\cos(2n\pi x) + \frac{3}{2n\pi}\sin(2n\pi x)\right]$$

(c) At $x = 1/2$, $f(\frac{1}{2}) = \frac{1}{2}[\ f(\frac{1}{2}+0) + f(\frac{1}{2}-0)]$

$= \frac{1}{2}[0.5 + 1.5]$

$= 1$

At $x = 2$, $f(2) = \frac{1}{2}[\ f(2+0) + f(2-0)]$

$= \frac{1}{2}[0.5 + 0]$

$= 0.25$

Q5) (a) $f(t) = \begin{cases} te^{-2t}, & 0 \le t < 1; \\ 0, & t \ge 1 \end{cases}$

$f(t) = u(t)\,te^{-2t} - u(t-1)\,te^{-2t}$

$= u(t)te^{-2t} - u(t-1)(t-1+1)e^{-2(t-1+1)}$

$= u(t)te^{-2t} - e^{-2}u(t-1)(t-1)e^{-2(t-1)} - e^{-2}u(t-1)e^{-2(t-1)}$

$F(s) = L\{f(t)\}$

$= L\{u(t)te^{-2t} - e^{-2}u(t-1)(t-1)e^{-2(t-1)} - e^{-2}u(t-1)e^{-2(t-1)}\}$

$= \frac{1}{(s+2)^2} - e^{-s}(e^{-2})(\frac{1}{(s+2)^2}) - e^{-s}(e^{-2})(\frac{1}{s+2})$

$= \frac{1}{(s+2)^2} - e^{-s-2}[\frac{1}{(s+2)^2} + \frac{1}{s+2}]$

(b) $L^{-1}\{\frac{e^{-s}}{s^2-8s+17}\}$

$f(t) = L^{-1}\{\frac{1}{s^2-8s+17}\}$

$= L^{-1}\{\frac{1}{(s-4)^2+1}\}$

$= e^{4t}L^{-1}\{\frac{1}{s^2+1}\}$

$= e^{4t}\sin t$

Hence, $L^{-1}\{\frac{e^{-s}}{s^2-8s+17}\}$

$= u(t-1)f(t-1)$

$= u(t-1)e^{4(t-1)}\sin(t-1)$

(c) $y' - y = \begin{cases} 0, & 0 \le t \le 2\pi \\ \sin t, & 2\pi < t \le 3\pi \\ 0, & t > 3\pi \end{cases}$ $\quad y(0) = 0$

Hence, $y' - y = u(t-2\pi)\sin t - u(t-3\pi)\sin t$

$= u(t-2\pi)\sin(t-2\pi) - u(t-3\pi)\sin(t-3\pi)(-1)$

$= u(t-2\pi)\sin(t-2\pi) + u(t-3\pi)\sin(t-3\pi)$

We now take laplace transform on both sides.

$L\{y'-y\} = L\{u(t-2\pi)\sin(t-2\pi) + u(t-3\pi)\sin(t-3\pi)\}$

$sL(y) - y(0) - L(y) = e^{-2\pi s}\left(\frac{1}{s^2+1}\right) + e^{-3\pi s}\left(\frac{1}{s^2+1}\right)$

let $Y = L(y)$,

$sY - 0 - Y = e^{-2\pi s}\left(\frac{1}{s^2+1}\right) + e^{-3\pi s}\left(\frac{1}{s^2+1}\right)$

$Y = e^{-2\pi s}\left(\frac{1}{(s^2+1)(s-1)}\right) + e^{-3\pi s}\left(\frac{1}{(s^2+1)(s-1)}\right)$ —①

$\frac{1}{(s^2+1)(s-1)} = \frac{As+C}{s^2+1} + \frac{B}{s-1}$

$= \frac{As^2 + Cs - As - C + Bs^2 + B}{(s^2+1)(s-1)}$

By comparing coefficient, $\left.\begin{array}{l} A+B=0 \\ -A+C=0 \\ B-C=1 \end{array}\right\}$ solving, $A = -\frac{1}{2}$, $B = \frac{1}{2}$, $C = -\frac{1}{2}$

From ①, $Y = e^{-2\pi s}\left(\frac{1}{2}\left(\frac{1}{s-1}\right) - \frac{1}{2}\left(\frac{s+1}{s^2+1}\right)\right) + e^{-3\pi s}\left(\frac{1}{2}\left(\frac{1}{s-1}\right) - \frac{1}{2}\left(\frac{s+1}{s^2+1}\right)\right)$

$= \frac{1}{2}e^{-2\pi s}\left(\frac{1}{s-1} - \frac{s}{s^2+1} - \frac{1}{s^2+1}\right) + \frac{1}{2}e^{-3\pi s}\left(\frac{1}{s-1} - \frac{s}{s^2+1} - \frac{1}{s^2+1}\right)$

Hence, $y(t) = L^{-1}\{Y\}$

$= L^{-1}\left\{\frac{1}{2}e^{-2\pi s}\left(\frac{1}{s-1} - \frac{s}{s^2+1} - \frac{1}{s^2+1}\right) + \frac{1}{2}e^{-3\pi s}\left(\frac{1}{s-1} - \frac{s}{s^2+1} - \frac{1}{s^2+1}\right)\right\}$

$= \frac{1}{2}u(t-2\pi)[e^{t-2\pi} - \cos(t-2\pi) - \sin(t-2\pi)]$

$+ \frac{1}{2}u(t-3\pi)[e^{t-3\pi} - \cos(t-3\pi) - \sin(t-3\pi)]$

$= \frac{1}{2}u(t-2\pi)[e^{(t-2\pi)} - \cos t - \sin t] + \frac{1}{2}u(t-3\pi)[e^{(t-3\pi)} + \cos t + \sin t]$

NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER 1 EXAMINATION 2018-2019

MA2006 – ENGINEERING MATHEMATICS

November/December 2018 Time Allowed: 2½ hours

INSTRUCTIONS

1. This paper contains **FOUR (4)** questions and comprises **SIX (6)** pages.
2. Answer **ALL FOUR** questions.
3. All questions carry equal marks.
4. This is a **CLOSED BOOK** examination.
5. Mathematical tables and formulae are provided on pages 4 to 6.

---

1 (a) By evaluating the determinant of a 4×4 matrix, determine the value of the constant k such that the homogeneous system of linear algebraic equations given by

$$\begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & 1 & 1 & 0 \\ 0 & 3 & 1 & 0 \\ 1 & k & 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

has infinitely many solutions.

(9 marks)

(b) For the k value you obtain in part 1(a), solve the above system of linear algebraic equations.

(9 marks)

(c) For a $n\times n$ matrix A, given: $A^2 = A$. Find all the possible eigenvalues of matrix A.

(7 marks)

34

2 (a) If $\underline{\mathbf{F}}$ is a differentiable vector function of one variable u, prove or disprove that $\underline{\mathbf{F}} \bullet \dfrac{d\underline{\mathbf{F}}}{du} = |\underline{\mathbf{F}}| \dfrac{d}{du}(|\underline{\mathbf{F}}|)$. (**Note.** Let $\underline{\mathbf{F}} = f(u)\underline{\mathbf{i}} + g(u)\underline{\mathbf{j}} + h(u)\underline{\mathbf{k}}$.)

(5 marks)

In all the parts below, Cartesian coordinates are denoted by the usual symbols x, y and z.

(b) Points (x, y, z) on the curve C are given by $x = 4 + t^2$, $y = t^2 + t + 1$ and $z = t$. Evaluate each of the following line integrals over C.

(i) $\int_C (16z - z^2 + x)\ ds$ for $1 \le t \le 2$.

(5 marks)

(ii) $\int_C (z\underline{\mathbf{i}} + y\underline{\mathbf{k}}) \bullet d\underline{\mathbf{r}}$, if $\underline{\mathbf{r}} = x\underline{\mathbf{i}} + y\underline{\mathbf{j}} + z\underline{\mathbf{k}}$ and C is assigned the direction from $(8, 7, 2)$ to $(5, 3, 1)$.

(5 marks)

(c) The surface S is the portion of the plane $x + y + z = 1$ in the first quadrant of the $Oxyz$ Cartesian coordinate space. The directed straight lines from $(1, 0, 0)$ to $(0, 0, 1)$, $(0, 0, 1)$ to $(0, 1, 0)$ and $(0, 1, 0)$ back to $(1, 0, 0)$ are denoted by C_1, C_2 and C_3 respectively. The union of C_1, C_2 and C_3 form a closed path C with a particular direction. Use Stokes' theorem to evaluate the line integral $\int_C (xy\,dx + z\,dy + zydz)$.

(Hint. The relation $\int_C \underline{\mathbf{F}} \bullet d\underline{\mathbf{r}} = \iint_S (\nabla \times \underline{\mathbf{F}}) \bullet \underline{\mathbf{n}}\ d\sigma$ is given in Stokes' theorem.)

(10 marks)

3 (a) A periodic function is defined in one period as

$$f(x)=\begin{cases} 2-\dfrac{4}{\pi}x, & 0\le x<\dfrac{\pi}{2}; \\ \dfrac{4}{\pi}x-3, & \dfrac{\pi}{2}\le x\le \pi. \end{cases}$$

(i) Sketch $f(x)$ over the interval $-\pi \le x < \pi$. (5 marks)

(ii) Find the Fourier series for $f(x)$. (10 marks)

(iii) Find the values of the Fourier series at $x = \pi$ and $3\pi/2$. (5 marks)

(b) If the function $f(x)$ in part 3(a) is not periodic but is only defined in the interval $0 \le x < \pi$, a Fourier cosine series can be obtained for $f(x)$ by using the even expansion. Sketch the even expansion for $f(x)$ in this case over the interval $-\pi \le x < 3\pi$.

(5 marks)

4 (a) Find the Laplace transform to the function $f(t)$ defined as

$$f(t)=\begin{cases} t^2, & 0\le t<1; \\ 0, & 1\le t<\pi; \\ e^{-t}\sin(t), & \pi\le t<2\pi; \\ 0, & t>2\pi. \end{cases}$$

(9 marks)

(b) Find the inverse Laplace transform

$$L^{-1}\left\{\frac{s^2-3s+4}{(s-1)(s^2-4s+5)}\right\}.$$

(6 marks)

(c) Solve the following ordinary differential equation for $y(t)$ by using the Laplace transform.

$$y'+2y=\begin{cases} 0, & 0\le t\le 2; \\ e^{-t}, & 2<t\le 3; \\ 0, & t>3 \end{cases} \qquad y(0)=0.$$

(10 marks)

FORMULAE FOR VECTOR CALCULUS PART

1. $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}, \quad \vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k},$

   $\vec{a} \bullet \vec{b} = a_1b_1 + a_2b_2 + a_3b_3,$

   $$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - b_2a_3)\vec{i} + (a_3b_1 - b_3a_1)\vec{j} + (a_1b_2 - b_1a_2)\vec{k}.$$

2. $grad\ f = \nabla f = \dfrac{\partial f}{\partial x}\vec{i} + \dfrac{\partial f}{\partial y}\vec{j} + \dfrac{\partial f}{\partial z}\vec{k},$ for scalar function $f(x, y, z)$.

3. $\vec{V} = p(x,y,z)\vec{i} + q(x,y,z)\vec{j} + r(x,y,z)\vec{k},$

   $\nabla \bullet \vec{V} = div\vec{V} = \dfrac{\partial p}{\partial x} + \dfrac{\partial q}{\partial y} + \dfrac{\partial r}{\partial z},$

   $$\nabla \times \vec{V} = curl\ \vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ p & q & r \end{vmatrix} = \left(\frac{\partial r}{\partial y} - \frac{\partial q}{\partial z}\right)\vec{i} + \left(\frac{\partial p}{\partial z} - \frac{\partial r}{\partial x}\right)\vec{j} + \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right)\vec{k},$$

4. $ds = \sqrt{(dx)^2 + (dy)^2 + (dz)^2} = \sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2 + \left(\dfrac{dz}{dt}\right)^2}\, dt,$

   $$\int_C f(x,y,z)ds = \int_{t_1}^{t_2} f[x(t), y(t), z(t)]\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt.$$

5. $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}, \quad d\vec{r} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k},$ For $\vec{F} = p(x,y,z)\vec{i} + q(x,y,z)\vec{j} + r(x,y,z)\vec{k},$

   $$\int_C \vec{F} \bullet d\vec{r} = \int p\,dx + q\,dy + r\,dz = \int_{t_1}^{t_2}\left(p\frac{dx}{dt} + q\frac{dy}{dt} + r\frac{dz}{dt}\right)dt.$$

6. Green's Theorem

   $$\oint_C f(x,y)dx + g(x,y)dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)dA$$

7. Surface integral:

   $$\iint_S g(x,y,z)d\sigma = \iint_R g[x,y,z(x,y)]\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dA,$$ for the surface given by $z = f(x,y)$.

FORMULAE FOR SPECIAL FUNCTIONS

$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$ $\quad$ $\sin x \sin y = [-\cos(x+y) + \cos(x-y)]/2$

$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$ $\quad$ $\cos x \cos y = [\cos(x+y) + \cos(x-y)]/2$

$\sin x \cos y = [\sin(x+y) + \sin(x-y)]/2$

$\sin^2 x = (1 - \cos 2x)/2, \quad \cos^2 x = (1 + \cos 2x)/2$

$\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}, \quad \sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}$

$\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}, \quad \cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$

$\sinh x = (e^x - e^{-x})/2, \quad \cosh x = (e^x + e^{-x})/2, \ \sinh x = -i \sin ix, \quad \cosh x = \cos ix$

FORMULAE FOR FOURIER SERIES

Euler Formulae for Fourier series for a periodic function $f(x)$ with a period $P = 2L$

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L})$$

where $a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx, \ a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \ b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx$

Complex form of Fourier series for $f(x)$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L} \quad \text{where} \quad c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}dx, \ n = 0, \pm 1, \pm 2, \cdots$$

LAPLACE TRANSFORM TABLE

$f(t) = L^{-1}\{F(s)\}$	$F(s) = L\{f(t)\}$	$f(t) = L^{-1}\{F(s)\}$	$F(s) = L\{f(t)\}$
1	$\frac{1}{s}$	$\frac{\sinh at}{a}$	$\frac{1}{s^2 - a^2}$
$\frac{t^{n-1}}{(n-1)!}$	$\frac{1}{s^n}, \ (n = 1, 2, \cdots)$	$\cosh at$	$\frac{s}{s^2 - a^2}$
$\frac{t^{n-1}e^{at}}{(n-1)!}$	$\frac{1}{(s-a)^n}, \ (n = 1, 2, \cdots)$	$u(t-a)$, Unit step function	$\frac{e^{-as}}{s}$
$\frac{\sin \omega t}{\omega}$	$\frac{1}{s^2 + \omega^2}$	$\delta(t-a)$, Unit impulse function	e^{-as}
$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$		

36

LAPLACE TRANSFORM FORMULAE

$f(t)=L^{-1}\{F(s)\}$	$F(s)=L\{f(t)\}$	Remarks
$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-\cdots-f^{(n-1)}(0)$	Differentiation of function
$\int_0^t f(\tau)d\tau$	$F(s)/s$	Integration of function
$e^{at}f(t)$	$F(s-a)$	Shift on s-axis
$u(t-a)f(t-a)$	$e^{-as}F(s)$	Shift on t-axis
$tf(t)$	$-F'(s)=-dF/ds$	Differentiation of transform
$t^nf(t)$	$(-1)^nF^{(n)}(s)$	Differentiation of transform
$f(t)/t$	$\int_s^\infty F(\tilde{s})d\tilde{s}$	Integration of transform
$\int_0^t f(t-\tau)g(\tau)d\tau$ $=\int_0^t f(\tau)g(t-\tau)d\tau$	$L(f*g)=L(f)L(g)$	Convolution
$f(t)=f(t+p)$	$\int_0^p e^{-st}f(t)dt\Big/(1-e^{-sp})$	$f(t)$ is periodic with period p

END OF PAPER

MA 2006 ENGINEERING MATHEMATICS (SEMESTER 1 EXAMINATION 2018-19)

1) (a)

$$\det\begin{pmatrix}1&2&1&1\\2&1&1&0\\0&3&1&0\\1&k&2&3\end{pmatrix} = \begin{vmatrix}1&1&0\\3&1&0\\k&2&3\end{vmatrix} - 2\begin{vmatrix}2&1&1\\3&1&0\\k&2&3\end{vmatrix} - 1\begin{vmatrix}2&1&1\\1&1&0\\3&1&0\end{vmatrix}$$

$$= 3(1-3) - 2\left(1\begin{vmatrix}3&1\\k&2\end{vmatrix} + 3\begin{vmatrix}2&1\\3&1\end{vmatrix}\right) - 1(1-3)$$

$$= -6 - 2(6-k+3(2-3)) + 2 = -10+2k$$

When determinant $=0$, we have infinitely many solution.

$$0 = -10+2k$$

$$k=5$$

(b)
$$\begin{pmatrix}1&2&1&1\\2&1&1&0\\0&3&1&0\\1&5&2&3\end{pmatrix} \rightarrow \begin{pmatrix}1&2&1&1\\0&-4&-3&-6\\0&3&1&0\\0&3&1&2\end{pmatrix} \rightarrow \begin{pmatrix}1&2&1&1\\0&1&1/3&2/3\\0&0&0&-2\\0&3&1&2\end{pmatrix} \rightarrow \begin{pmatrix}1&2&1&1\\0&1&1/3&2/3\\0&0&0&-2\\0&0&0&0\end{pmatrix}$$

$$\left(\begin{array}{cccc|c}1&2&1&1&0\\0&1&1/3&2/3&0\\0&0&0&-2&0\\0&0&0&0&0\end{array}\right)$$

Let $x = s$ (Any arbitrary number)

$-2w=0$, $w=0$

$y+\frac{1}{3}z=0$, $y=\frac{1}{3}z = -\frac{1}{5}s$

$s+2y+z=0$, $z=-\frac{3}{5}s$

$$\begin{pmatrix}x\\y\\z\\w\end{pmatrix} = \begin{pmatrix}s\\-\frac{1}{5}s\\-\frac{3}{5}s\\0\end{pmatrix}$$

~~[illegible]~~

(c) Square matrix that has property $A^2=A$ is idempotent.

Let A = Idempotent square matrix

λ = Eigenvalue

v = Corresponding eigenvector

$$\lambda v = Av = AAv = \lambda A v = \lambda^2 v$$

$$\lambda v - \lambda^2 v = 0$$

$$\lambda(1-\lambda) = 0$$

$\lambda = 0$ or $\lambda = 1$

(1) 4

2) (a) $\underset{\sim}{F} = f(u)\underset{\sim}{i} + g(u)\underset{\sim}{j} + h(u)\underset{\sim}{k}$

$\frac{dF}{du} = \frac{df}{du}\underset{\sim}{i} + \frac{dg}{du}\underset{\sim}{j} + \frac{dh}{du}\underset{\sim}{k}$

$\underset{\sim}{F} \cdot \frac{dF}{du} = \begin{pmatrix} f(u) \\ g(u) \\ h(u) \end{pmatrix} \begin{pmatrix} \frac{df}{du} \\ \frac{dg}{du} \\ \frac{dh}{du} \end{pmatrix} = f(u)\frac{df}{du} + g(u)\frac{dg}{du} + h(u)\frac{dh}{du}$ #

$|\underset{\sim}{F}| \frac{d}{du}(|\underset{\sim}{F}|) = \sqrt{f^2(u) + g^2(u) + h^2(u)} \quad \frac{d}{du}\left(\sqrt{f(u)^2 + g(u)^2 + h(u)^2}\right)$

$= \sqrt{f(u)^2 + g(u)^2 + h(u)^2} \left[\frac{1}{2}\left(f(u)^2 + g(u)^2 + h(u)^2\right)^{-\frac{1}{2}} \left(2f(u)\frac{df}{du} + 2g(u)\frac{dg}{du} + 2h(u)\frac{dh}{du}\right)\right]$

$= f(u)\frac{df}{du} + g(u)\frac{dg}{du} + h(u)\frac{dh}{du}$ #

(b)(i) $\frac{dx}{dt} = 2t \quad \frac{dy}{dt} = 2t+1 \quad \frac{dz}{dt} = 1$

$\int_C (16z - z^2 + x)\, ds = \int_1^2 (16t - t^2 + 4 + t^2)\sqrt{(2t)^2 + (2t+1)^2 + (1)^2}\, dt$

$= \int_1^2 (16t + 4)\sqrt{8t^2 + 4t + 2}\, dt$ Let $u = 8t^2 + 4t + 2$, $du = 16t + 4\, dt$

$= \int_{14}^{42} \sqrt{u}\, du = \left[\frac{u^{\frac{3}{2}}}{3/2}\right]_{14}^{42} = \frac{2}{3}\left(42^{3/2} - 14^{3/2}\right)$ #

(ii) $\int_C \begin{pmatrix} z \\ 0 \\ y \end{pmatrix} \cdot d\underset{\sim}{r} = \int_2^1 \begin{pmatrix} t \\ 0 \\ t^2+t+1 \end{pmatrix} \cdot \begin{pmatrix} 2t \\ 2t+1 \\ 1 \end{pmatrix} dt$

$= \int_2^1 2t^2 + t^2 + t + 1\, dt = \left[t^3 + \frac{t^2}{2} + t\right]_2^1 = -9.5$ #

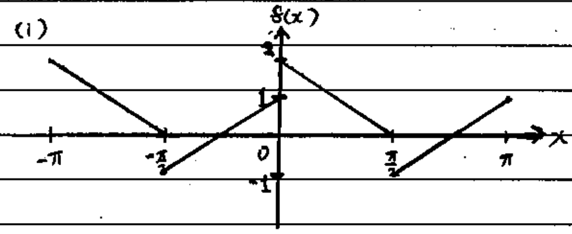
DISCLAIMER: *The solutions are done by students who scored A or above in this subject. MAE Club and Campus supplies are not liable or responsible for any errors in the contents of these solutions. Students are advised to take the solutions as a guide rather than absolute answers to exam paper.*

Should there be any mistake identified, please proceed to the Facebook link encoded in the QR code to feedback or submit correct answers. The link is: http://bit.ly/2lW2C32

3)(a)

$$f(x) = \begin{cases} 2 - \frac{4}{\pi}x, & 0 \le x < \frac{\pi}{2} \\ \frac{4}{\pi}x - 3, & \frac{\pi}{2} \le x \le \pi \end{cases}$$

(i)



(ii) $f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$ $\quad P = 2L = \pi$

$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$, $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$, $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$

$$a_0 = \frac{1}{\pi}\left[\int_0^{\pi/2} 2 - \frac{4}{\pi}x\,dx + \int_{\pi/2}^{\pi} \frac{4}{\pi}x - 3\,dx\right]$$

$$= \frac{1}{\pi}\left[\left(2x - \frac{2}{\pi}x^2\right)_0^{\pi/2} + \left(\frac{2}{\pi}x^2 - 3x\right)_{\pi/2}^{\pi}\right] = \frac{1}{2}$$ //

$$a_n = \frac{2}{\pi}\int_0^{\pi/2}\left(2 - \frac{4}{\pi}x\right)\cos(2nx)\,dx + \frac{2}{\pi}\int_{\pi/2}^{\pi}\left(\frac{4}{\pi}x - 3\right)\cos(2nx)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi/2} 2\cos(2nx) - \frac{4}{\pi}x\cos(2nx)\,dx + \frac{2}{\pi}\int_{\pi/2}^{\pi}\frac{4}{\pi}x\cos(2nx) - 3\cos(2nx)\,dx$$

$$= \frac{2}{\pi}\left[\frac{2\sin(2nx)}{2n}\right]_0^{\pi/2} - \frac{8}{\pi^2}\left[\frac{x\sin(2nx)}{2n} - \left[\frac{-\cos(2nx)}{4n^2}\right]\right]_0^{\pi/2} + \frac{8}{\pi^2}\left[\frac{x\sin(2nx)}{2n} - \left[-\frac{\cos(2nx)}{4n^2}\right]\right]_{\pi/2}^{\pi}$$

$$- \frac{6}{\pi}\left[\frac{\sin(2nx)}{2n}\right]_{\pi/2}^{\pi}$$

$$= -\frac{8}{\pi^2}\left[\frac{(-1)^n}{4n^2} - \frac{1}{4n^2}\right] + \frac{8}{\pi^2}\left[\frac{1}{4n^2} - \frac{(-1)^n}{4n^2}\right] = \frac{4}{n^2\pi^2} - \frac{4(-1)^n}{n^2\pi^2} = \frac{4}{n^2\pi^2}\left(1 - (-1)^n\right)$$ //

$$b_n = \frac{2}{\pi}\int_0^{\pi/2}\left(2 - \frac{4}{\pi}x\right)\sin(2nx)\,dx + \frac{2}{\pi}\int_{\pi/2}^{\pi}\left(\frac{4}{3}x - 3\right)\sin(2nx)\,dx$$

$$= \frac{4}{\pi}\int_0^{\pi/2}\sin(2nx) - \frac{8}{\pi^2}\int_0^{\pi/2} x\sin(2nx)\,dx + \frac{8}{3\pi}\int_{\pi/2}^{\pi} x\sin(2nx)\,dx - \frac{6}{\pi}\int_{\pi/2}^{\pi}\sin(2nx)\,dx$$

$$= \frac{4}{\pi}\left[\frac{-\cos(2nx)}{2n}\right]_0^{\pi/2} - \frac{8}{\pi^2}\left[\frac{-x\cos(2nx)}{2n} - \left[-\frac{\sin(2nx)}{4n^2}\right]\right]_0^{\pi/2} + \frac{8}{3\pi}\left[\frac{-x\cos(2nx)}{2n} - \left[-\frac{\sin(2nx)}{4n^2}\right]\right]_{\pi/2}^{\pi} - \frac{6}{\pi}\left[\frac{-\cos(2nx)}{2n}\right]_{\pi/2}^{\pi}$$

$$= \frac{4}{\pi}\left[-\frac{(-1)^n}{2n} + \frac{1}{2n}\right] - \frac{8}{\pi^2}\left[\frac{-\frac{\pi}{2}(-1)^n}{2n}\right] + \frac{8}{3\pi}\left[\frac{-\pi}{2n} + \frac{\frac{\pi}{2}(-1)^n}{2n}\right] + \frac{6}{\pi}\left[\frac{1}{2n} - \frac{(-1)^n}{2n}\right]$$

$$= \frac{5}{n\pi}\left[1 - (-1)^n\right] - \frac{4}{3n} + \frac{2}{3}(-1)^n + \frac{2}{\pi}(-1)^n$$ //

DISCLAIMER: *The solutions are done by students who scored A or above in this subject. MAE Club and Campus supplies are not liable or responsible for any errors in the contents of these solutions. Students are advised to take the solutions as a guide rather than absolute answers to exam paper.*

Should there be any mistake identified, please proceed to the Facebook link encoded in the QR code to feedback or submit correct answers. The link is: http://bit.ly/2lW2C32

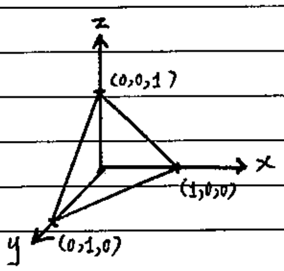
(2) //

(c) $x+y+z=1$

$\vec{n} = (1,1,1)$

$\hat{n} = \frac{(1,1,1)}{\sqrt{1^2+1^2+1^2}}$

$= \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$



$\text{curl}(\vec{F}) = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \times \begin{pmatrix} xy \\ z \\ zy \end{pmatrix}$

$= (z-1)\underset{\sim}{i} - 0\underset{\sim}{j} + (-x)\underset{\sim}{k}$

$= (z-1)\underset{\sim}{i} - x\underset{\sim}{k}$

$\int_c (xy\, \delta x + z\, \delta y + zy\, \delta z)$

$= \iint_S \begin{pmatrix} z-1 \\ 0 \\ -x \end{pmatrix} \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} \sqrt{3}\, \delta A$

$= \iint_S z-1-x \quad \delta A$

$= \int_0^1 \int_0^{1-y} [(1-x-y)-1-x]\, \delta x\, \delta y$

$= \int_0^1 \left[-x^2-yx\right]_0^{1-y} \delta y$

$= \int_0^1 y-1\, \delta y$

$= \left[\frac{y^2}{2} - y\right]_0^1$

$= -\frac{1}{2}$ //

$\delta\sigma = \sqrt{1+\left(\frac{df}{dx}\right)^2+\left(\frac{df}{dy}\right)^2} \qquad f = z = 1-x-y$

$= \sqrt{1+(-1)^2+(-1)^2}$

$= \sqrt{3}\ \delta A$

DISCLAIMER: *The solutions are done by students who scored A or above in this subject. MAE Club and Campus supplies are not liable or responsible for any errors in the contents of these solutions. Students are advised to take the solutions as a guide rather than absolute answers to exam paper.*

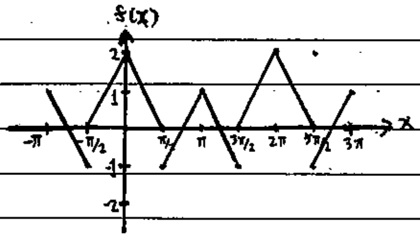
Should there be any mistake identified, please proceed to the Facebook link encoded in the QR code to feedback or submit correct answers. The link is: http://bit.ly/2lW2C32

$\therefore f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{4}{n^2\pi^2}(1-(-1)^n)\cos(2nx) + \left[\frac{5}{n\pi}(1-(-1)^n) - \frac{4}{3n} + \left(\frac{2}{3}+\frac{2}{\pi}\right)(-1)^n \right] \sin(2nx) \right)$ #

(iii) When $x = \pi$, $f(x) = 1.5$ #

When $x = 3\pi/2$, $f(x) = -0.5$ #

(b)



4 (a) $f(t) = t^2 - u(t-1)t^2 + u(t-\pi)e^{-t}\sin(t) - u(t-2\pi)e^{-t}\sin(t)$

$= t^2 - u(t-1)(t-1+1)^2 + u(t-\pi)e^{-(t-\pi)}[-\sin(t-\pi)]e^{-\pi} - u(t-2\pi)e^{-(t-2\pi)}\sin(t-2\pi)e^{-2\pi}$

$= t^2 - u(t-1)((t-1)^2+2(t-1)+1) - u(t-\pi)e^{-(t-\pi)}\sin(t-\pi)e^{-\pi} - u(t-2\pi)e^{-(t-2\pi)}\sin(t-2\pi)e^{-2\pi}$

$F(s) = \frac{2}{s^3} - e^{-s}\left(\frac{2}{s^3}\right) - 2e^{-s}\left(\frac{1}{s^2}\right) - e^{-s}\left(\frac{1}{s}\right) - e^{-\pi s}\frac{1}{(s+1)^2+1}e^{-\pi} - e^{-2\pi s}\frac{1}{(s+1)^2+1}e^{-2\pi}$ #

(b) $\frac{s^2-3s+4}{(s-1)(s^2-4s+5)} = \frac{A}{s-1} + \frac{Bs+C}{s^2-4s+5}$

$s^2-3s+4 = A(s^2-4s+5) + (Bs+C)(s-1)$

$= (A+B)s^2 + (C-B-4A)s + 5A - C$

$= \frac{1}{s-1} + \frac{1}{s^2-4s+5}$

$A+B = 1 \Rightarrow B = 1-A$ $\quad A = 1$

$C-B-4A = -3 \Rightarrow C = 3A-2$ $\quad B = 0$

$5A - C = 4 \Rightarrow C = 5A-4$ $\quad C = 1$

$= \frac{1}{s-1} + \frac{1}{(s-2)^2+1}$

$\therefore f(t) = e^{t} + e^{2t}\sin(t)$ #

$e^{-2s-2} - e^{-3s-3} = A(s+2) + B(s+1)$

$A = 0$; $B = e^3 - e^2$

(c) $y' + 2y = u(t-2)e^{-t} - u(t-3)e^{-t}$

$= u(t-2)e^{-(t-2)}e^{-2} - u(t-3)e^{-(t-3)}e^{-3}$

$sY + 2Y = e^{-2s}\frac{e^{-2}}{s+1} - e^{-3s}\frac{e^{-3}}{s+1}$

$Y(s+2) = \frac{1}{s+1}(e^{-2s-2} - e^{-3s-3})$

$Y = (e^{-2s-2} - e^{-3s-3})\frac{1}{(s+1)(s+2)} = \frac{e^3-e^2}{s+2}$

$y(t) = (e^3-e^2)e^{-2t}$ #

(3)#

NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER 2 EXAMINATION 2018-2019

MA2006 – ENGINEERING MATHEMATICS

April/May 2019 — Time Allowed: 2½ hours

INSTRUCTIONS

1. This paper contains **FOUR (4)** questions and comprises **SIX (6)** pages.
2. Answer **ALL** questions.
3. All questions carry equal marks.
4. This is a **CLOSED-BOOK** examination.
5. Mathematical formulae and tables are provided on pages 4, 5 and 6.

1(a) For $n \times n$ matrices A, B, C and I (here I is a $n \times n$, identity matrix), given:

$$B = I + AB \quad \text{and} \quad C = A + CA.$$

Prove: $B - C = I$.

(8 marks)

(b) For different values of k, discuss the possible solution cases of the following linear system (i.e., under what values of k, the system has unique solution, no solution or infinitely many solutions, respectively).

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 2x_1 + 3x_2 + (k+2)\,x_3 &= 3 \\ x_1 + k\,x_2 - 2x_3 &= 0 \end{aligned}$$

(8 marks)

(c) If the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & a \end{bmatrix}$$

has an eigenvalue 0, find all the eigenvalues and eigenvectors of matrix A.

(9 marks)

2. In all the parts below, Cartesian coordinates are denoted by the usual symbols x, y and z.

(a) Evaluate each of the following integrals:

(i) $\int_C (x+4y+z)\, ds$, C is the straight line from $(1,0,2)$ to $(-1,1,0)$.

(8 marks)

(ii) $\iint_S (x+z)\, d\sigma$, S is the portion of the plane $2x+2y+z=10$ in the region

$$x^2+y^2 \le 1,\ -\infty < z < \infty.$$

(8 marks)

(b) Find the constant c such that the two-dimensional force field given by the vector function $\underline{F}(x,y) = (cxy+1)\underline{i} + (x^2+3y^2)\underline{j}$ is conservative. For the value of c which you obtain, calculate the work done by $\underline{F}$ along curve C, given that this curve C is defined by $x(t) = \sin(\pi t)$ and $y(t) = \cos(2\pi t)$ from $t=0$ to $t=1/2$.

(9 marks)

3(a) A periodic function is defined in one period as

$$f(x) = \begin{cases} 2, & 0 \le x \le 1; \\ 2-x, & 1 < x \le 3. \end{cases}$$

(i) Sketch $f(x)$ over the interval $-3 \le x < 3$.

(3 marks)

(ii) Find the Fourier series for $f(x)$.

(8 marks)

(iii) Find the values of the Fourier series at $x=3$ and $x=4$.

(4 marks)

(b) If the function $f(x)$ in 3(a) is not periodic but is only defined in the interval $0 \le x < 3$, a <u>Fourier sine series</u> can be obtained for $f(x)$ by a proper expansion.

(i) Sketch the expansion for $f(x)$ over the interval $-3 \le x \le 9$.

(5 marks)

(ii) Find the Fourier sine series for $f(x)$.

(5 marks)

40

4(a) Find the Laplace transform to the function $f(t)$ defined as

$$f(t)=\begin{cases} e^{-t}\cos(t), & 0\le t<\pi/2; \\ 0, & \pi/2\le t<2; \\ te^{-t}, & 2\le t<3; \\ 0, & t>3. \end{cases}$$

(9 marks)

(b) Find the inverse Laplace transform

$$L^{-1}\left\{\frac{2s^3-6s^2+22s+13}{s^2(s^2-4s+13)}\right\}.$$

(6 marks)

(c) Solve the following ordinary differential equation for $y(t)$ by using the Laplace transform.

$$y'+4y=\begin{cases} e^{-t}, & 0\le t\le 2; \\ 0, & t>2. \end{cases} \qquad y(0)=1.$$

(10 marks)

MA2006 18/19 Sem 2

1a) $B = I + AB \Rightarrow BB^{-1} = IB^{-1} + ABB^{-1} \Rightarrow I = B^{-1} + A$ — (1)

$C = A + CA \Rightarrow C = A(I+C) \Rightarrow A = C(I+C)^{-1}$ — (2)

subst (1) into (2):

$I = B^{-1} + C(I+C)^{-1} \Rightarrow I(I+C) = B^{-1}(I+C) + C(I+C)^{-1}(I+C)$

$\Rightarrow I + C = B^{-1}(I+C) + C \Rightarrow B(I+C) = BB^{-1}(I+C) + BC$

$\Rightarrow B + BC = I + C + BC \Rightarrow B = I + C \Rightarrow B - C = I$

1b) To have unique solution, the determinant from the coeff of the linear system $\neq 0$

$A = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & k+2 \\ 1 & k & -2 \end{vmatrix} \neq 0 \Rightarrow \det(A) = \det\begin{pmatrix} 3 & k+2 \\ k & -2 \end{pmatrix} + (-1)(2)\det\begin{pmatrix} 2 & k+2 \\ 1 & -2 \end{pmatrix} + \det\begin{pmatrix} 2 & 3 \\ 1 & k \end{pmatrix}$

$= -6 - k^2 - 2k - 2(-4 - k - 2) + 2k - 3$

$= -6 - k^2 - 2k + 12 + 2k + 2k - 3$

$= -k^2 + 2k + 3 = -(k^2 - 2k - 3)$

$= -(k-3)(k+1)$

<1> For unique solution, $k \neq 3$ and $k \neq -1$

<2> For non-unique solution, $k = 3$ and $k = -1$

↳ check whether infinite or no solution

↳ When $k = 3$

$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & 3 & 5 & 3 \\ 1 & 3 & -2 & 0 \end{array}\right] \begin{array}{l} \\ \to R_2 - 2R_1 \\ \to R_3 - R_1 \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & -1 & 3 & 1 \\ 0 & 1 & -3 & -1 \end{array}\right] \begin{array}{l} \\ \\ \to R_3 + R_2 \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & -1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

∴ infinite solution when $k = 3$

↳ When $k = -1$

$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & 3 & 1 & 3 \\ 1 & -1 & -2 & 0 \end{array}\right] \begin{array}{l} \\ \to R_2 - 2R_1 \\ \to R_3 - R_1 \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & -1 & -1 & 1 \\ 0 & -3 & -3 & -1 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 3R_2 \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & -4 \end{array}\right]$

∴ no solution when $k = -1$

1c) Given $A=\begin{pmatrix}1&0&1\\0&2&0\\1&0&a\end{pmatrix}$

The characteristic equation is

$\det\begin{pmatrix}1-\lambda&0&1\\0&2-\lambda&0\\1&0&a-\lambda\end{pmatrix}=0 \Rightarrow (1-\lambda)\det\begin{pmatrix}2-\lambda&0\\0&a-\lambda\end{pmatrix}+\det\begin{pmatrix}0&2-\lambda\\1&0\end{pmatrix}=0$

$=(1-\lambda)(2-\lambda)(a-\lambda)+-(2-\lambda)=0$

$=(2-\lambda)[(1-\lambda)(a-\lambda)-1]=0$

Given $\lambda=0$ is an eigenvalue

$2[(1)(a)-1]=0 \Rightarrow a=1$

$\therefore (2-\lambda)[(1-\lambda)^2-1]=0 \Rightarrow (2-\lambda)(\lambda)(\lambda-2)=0$

The eigenvalues of A are $\lambda=0$ and $\lambda=2$

when $\lambda=0$

$\left(\begin{array}{ccc|c}1&0&1&0\\0&2&0&0\\1&0&1&0\end{array}\right) \to R_1-R_3 \Rightarrow \left(\begin{array}{ccc|c}1&0&1&0\\0&2&0&0\\0&0&0&0\end{array}\right) \Rightarrow \begin{array}{l}y=0\\x+z=0\\x=-z\end{array} \Rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix}=t\begin{pmatrix}1\\0\\-1\end{pmatrix}$

when $\lambda=2$

$\left(\begin{array}{ccc|c}-1&0&1&0\\0&0&0&0\\1&0&-1&0\end{array}\right) \to R_1+R_3 \Rightarrow \left(\begin{array}{ccc|c}-1&0&1&0\\0&0&0&0\\0&0&0&0\end{array}\right) \Rightarrow \begin{array}{l}x-z=0\\x=z\end{array} \Rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix}=t\begin{pmatrix}1\\0\\1\end{pmatrix}+s\begin{pmatrix}0\\1\\0\end{pmatrix}$

$\therefore$ the possible eigenvectors of matrix A are:

$\begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}$ and $\begin{pmatrix}0\\1\\0\end{pmatrix}$ # $P=\begin{pmatrix}1&1&0\\0&0&1\\-1&1&0\end{pmatrix}$ # and $D=\begin{pmatrix}0&0&0\\0&2&0\\0&0&2\end{pmatrix}$ #

2 a) i) Straight line from $\vec{OA}=\begin{pmatrix}1\\0\\2\end{pmatrix}$ to $\vec{OB}=\begin{pmatrix}-1\\1\\0\end{pmatrix}$

$AB=\begin{pmatrix}-2\\1\\-2\end{pmatrix}$ $=\begin{pmatrix}1\\0\\2\end{pmatrix}+t\begin{pmatrix}-2\\1\\-2\end{pmatrix} \Rightarrow$

$x=1-2t \Rightarrow \frac{dx}{dt}=-2$

$y=t \Rightarrow \frac{dy}{dt}=1$

$z=2-2t \Rightarrow \frac{dz}{dt}=-2$

$ds=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt=\sqrt{(-2)^2+(1)^2+(-2)^2}=3dt$

$\int_C (x+4y+z)\,ds=\int_0^1 (1-2t+4t+2-2t)\,3\,dt$

$=9$ #

ii) Equation of the surface $2x+2y+z=10 \Rightarrow z=10-2x-2y$

$d\sigma=\sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}\,dA=\sqrt{1+(-2)^2+(-2)^2}\,dA=3\,dA$

$\iint_S (x+z)\,d\sigma=\iint_R (x+10-2x-2y)\,3\,dA=3\iint_R 10-x-2y\,dA$

$=3\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1} 10-r\cos\theta-2r\sin\theta\,dr\,d\theta=60\pi$ #

$f(x,y) = \frac{\partial \phi}{\partial x}$

$g(x,y) = \frac{\partial \phi}{\partial y}$

2b) $\underline{F} = (cxy+1)\,\underline{i} + (x^2+3y^2)\,\underline{j}$

$\frac{\partial}{\partial x}(x^2+3y^2) = 2x \qquad \frac{\partial}{\partial y}(cxy+1) = cx$

$\therefore c = 2$

$\frac{\partial \phi}{\partial x} = 2xy+1 \Rightarrow \qquad \phi = x^2y + x + F(y)$

$\frac{\partial \phi}{\partial y} = x^2+3y^2 \Rightarrow \qquad \phi = x^2y + 6y^3 + G(x)$

A possible potential function is $\phi(x,y) = x^2y + x + 6y^3$

$x(t) = \sin(\pi t) \qquad y(t) = \cos(2\pi t)$

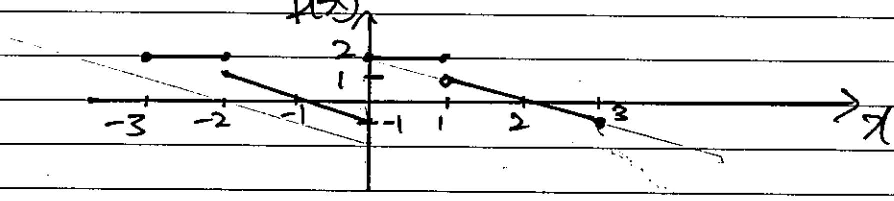
$t=0 \Rightarrow x=0 \qquad y=1$

$t=\frac{1}{2} \Rightarrow x=1 \qquad y=-1$

$\therefore$ Work done $= \phi(1,-1) - \phi(0,1)$

$= -12$ ✱

3a) i) $f(x) = \begin{cases} 2 & 0 \le x \le 1 \\ 2-x & 1 < x \le 3 \end{cases}$



ii) $f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right]$

$a_0 = \frac{1}{2L}\int_0^{2L} f(x)\,dx = \frac{1}{3}\int_0^3 f(x)\,dx = \frac{1}{3}(2) = \frac{2}{3}$

$a_n = \frac{1}{L}\int_0^{2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{3/2}\int_0^3 f(x)\cos\left(\frac{n\pi x}{3}\right)dx$

$= \frac{2}{3}\left[\int_0^1 2\cos\left(\frac{n\pi x}{3}\right)dx + \int_1^3 (2-x)\cos\left(\frac{n\pi x}{3}\right)dx\right]$

(next page)

iii) $\int 2\cos\left(\frac{n\pi x}{3}\right)dx$

$= 2\left(\frac{3}{n\pi}\right)\sin\left(\frac{n\pi x}{3}\right)+C$

$\int x\cos\left(\frac{n\pi x}{3}\right)dx$

$= x\left(\frac{3}{n\pi}\right)\sin\left(\frac{n\pi x}{3}\right) - \int\left(\frac{3}{n\pi}\right)\sin\left(\frac{n\pi x}{3}\right)dx$

$= \frac{3x}{n\pi}\sin\left(\frac{n\pi x}{3}\right) + \left(\frac{3}{n\pi}\right)^2\cos\left(\frac{n\pi x}{3}\right) + C$

$a_n = \frac{6}{n\pi}\left[\sin\left(\frac{n\pi x}{3}\right)\right]_0^1 + \frac{6}{n\pi}\left[\sin\left(\frac{n\pi x}{3}\right)\right]_1^3$ $\frac{6}{n\pi}\left[\sin\left(\frac{n\pi x}{3}\right)\right.$

$-\left[\frac{3x}{n\pi}\sin\left(\frac{n\pi x}{3}\right)+\left(\frac{3}{n\pi}\right)^2\cos\left(\frac{n\pi x}{3}\right)\right]_1^3$

$a_n = -\frac{2}{3}\left[\frac{9}{n\pi}\sin(3n\pi) + \left(\frac{3}{n\pi}\right)^2\cos(3\pi n) - \frac{3}{n\pi}\sin\left(\frac{n\pi}{3}\right) - \left(\frac{3}{n\pi}\right)^2\cos\left(\frac{n\pi}{3}\right)\right]$

$= \left[\frac{3}{n\pi}\sin\left(\frac{n\pi}{3}\right) + \left(\frac{3}{n\pi}\right)^2\cos\left(\frac{n\pi}{3}\right) - \frac{3}{n\pi}(-1)^n\right]\frac{2}{3}$

$b_n = \frac{1}{L}\int_0^{2L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$

$= \frac{1}{3/2}\int_0^3 f(x)\sin\left(\frac{n\pi x}{3}\right)dx$

$= \frac{2}{3}\left[\int_0^1 2\sin\left(\frac{n\pi x}{3}\right) + \int_1^3 (2-x)\sin\left(\frac{n\pi x}{3}\right)dx\right]$

$= \frac{2}{3}\left[\left[\frac{-6}{n\pi}\cos\left(\frac{n\pi x}{3}\right)\right]_0^3 + \left[\frac{3x}{n\pi}\cos\left(\frac{n\pi x}{3}\right) - \left(\frac{3}{n\pi}\right)^2\sin\left(\frac{n\pi}{3}\right)\right]_1^3\right.$

$= \frac{2}{3}\left[-\frac{6}{n\pi}\left[(-1)^n - 1\right] + \left[\frac{9}{n\pi}(-1)^n - \frac{3}{n\pi}\cos\left(\frac{n\pi}{3}\right) + \left(\frac{3}{n\pi}\right)^2\sin\left(\frac{n\pi}{3}\right)\right]\right]$

$= \frac{2}{3}\left[\frac{3(-1)^n}{n\pi} + \frac{6}{n\pi} - \frac{3}{n\pi}\cos\left(\frac{n\pi}{3}\right) + \left(\frac{3}{n\pi}\right)^2\sin\left(\frac{n\pi}{3}\right)\right]$

$\int 2\sin\left(\frac{n\pi x}{3}\right)dx$

$= -2\left(\frac{3}{n\pi}\right)\cos\left(\frac{n\pi x}{3}\right)+C$

$\int x\sin\left(\frac{n\pi x}{3}\right)dx$

$= -x\left(\frac{3}{n\pi}\right)\cos\left(\frac{n\pi x}{3}\right) - \int -\left(\frac{3}{n\pi}\right)\cos\left(\frac{n\pi x}{3}\right)dx$

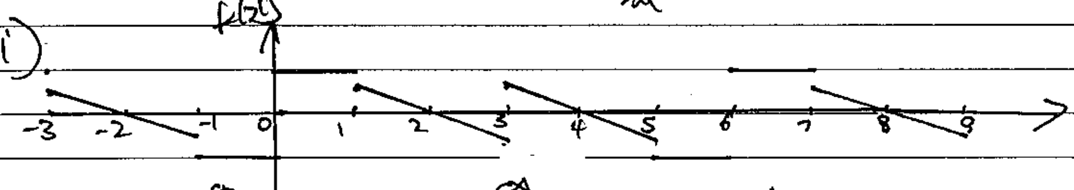
$= -x\left(\frac{3}{n\pi}\right)\cos\left(\frac{n\pi x}{3}\right) + \left(\frac{3}{n\pi}\right)^2\sin\left(\frac{n\pi x}{3}\right)+C$

iii) Based on piecewise continuous theorem

when $x=3$, $f(x) = \frac{-1+2}{2} = \frac{1}{2}$

when $x=4$, $f(x) = \frac{2+1}{2} = \frac{3}{2}$

b) i)



ii) $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{3}x\right)$

$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{3}\int_0^3 f(x)\sin\left(\frac{n\pi x}{3}\right)dx$

$= \frac{2}{3}\left[\frac{3(-1)^n}{n\pi} + \frac{6}{n\pi} - \frac{3}{n\pi}\cos\left(\frac{n\pi}{3}\right) + \left(\frac{3}{n\pi}\right)^2\sin\left(\frac{n\pi}{3}\right)\right]$

$\cos(t) = \sin(\frac{\pi}{2} - t)$
$= -\sin(t - \frac{\pi}{2})$

4a) $f(t) = e^{-t}\cos(t) - e^{-t}\cos(t)\,u(t-\frac{\pi}{2}) + te^{-t}\,u(t-2) - te^{-t}\,u(t-3)$

$= e^{-t}\cos(t)u(t) + e^{-\pi/2}e^{-(t-\frac{\pi}{2})}\sin(t-\frac{\pi}{2})\,u(t-\frac{\pi}{2})$
$+ (t-2)e^{-(t-2)}u(t-2)\cdot e^{-2} + \frac{2}{e^2}e^{-(t-2)}u(t-2)$
$- (t-3)e^{-(t-3)}u(t-3)\cdot e^{-3} - \frac{3}{e^3}e^{-(t-3)}u(t-3)$

$\mathcal{L}(f(t)) = \frac{s+1}{(s+1)^2+1} + e^{-\frac{\pi}{2}s-\frac{\pi}{2}}\frac{1}{(s+1)^2+1}$
$+ e^{-2s-2}\left(\frac{1}{(s+1)^2}\right) + \frac{2}{e^2}e^{-2s}\left(\frac{1}{s+1}\right)$
$- e^{-3s-3}\left(\frac{1}{(s+1)^2}\right) - \frac{3}{e^3}e^{-3s}\left(\frac{1}{s+1}\right)$

$= \frac{s+1}{(s+1)^2+1} + \frac{e^{-\frac{\pi}{2}s-\frac{\pi}{2}}}{(s+1)^2+1} + e^{-2s-2}\left[\frac{1}{s+1} + \frac{1}{(s+1)^2}\right]$
$- e^{-3s-3}\left[\frac{1}{s+1} + \frac{1}{(s+1)^2}\right]$

4b) $\frac{2s^3-6s^2+22s+13}{s^2(s^2-4s+13)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2-4s+13}$

$2s^3-6s^2+22s+13 = As(s^2-4s+13)$
$+B(s^2-4s+13)$
$+(Cs+D)s^2$

Let $s=0$
$13 = 13B \Rightarrow B=1$
Let $s=1$
$31 = A(10)+B(10)+C+D$
$10A+C+D=21$
Let $s=-1$
$-17 = -18A+18B-C+D$
$-18A-C+D=-35$
Let $s=2$
$49 = 18A+9B+8C+4D$
$18A+8C+4D=40$

$A=2$
$B=1$
$C=0$
$D=1$

$\therefore$ we get: $\frac{2}{s}+\frac{1}{s^2}+\frac{1}{(s-2)^2+3^2}$

$L^{-1}\left[\frac{2}{s}+\frac{1}{s^2}+\frac{3}{(s-2)^2+3^2}\left(\frac{1}{3}\right)\right] = 2+t+\frac{1}{3}\cdot e^{2t}\sin 3t$
$= 2+t+\frac{e^{2t}}{3}\sin 3t$

4c) $y' + 4y = \begin{cases} e^{-t} & 0 \le t \le 2 \\ 0 & t > 2 \end{cases} \qquad y(0) = 1$

$y' + 4y = e^{-t} u(t) - e^{-t} u(t-2)$

$= e^{-t} u(t) - \frac{e^{-(t-2)}}{e^2} u(t-2)$

$L(e^{-t} u(t)) = \frac{1}{s+1}$

$\frac{1}{e^2} L(e^{-(t-2)} u(t-2)) = \frac{1}{e^2} \cdot e^{-2s} \cdot \frac{1}{s+1} = \frac{e^{-2s-2}}{s+1}$

Laplace entire function:

$L(y' + 4y) = L[e^{-t} u(t) - e^{-t} u(t-2)]$

$sY - y(0) + 4Y = \frac{1}{s+1} - \frac{e^{-2s-2}}{s+1}$

$(s+4)Y - 1 = \frac{1 - e^{-2s-2}}{s+1}$

$(s+4)Y = \frac{(s+1) + 1 - e^{-2s-2}}{s+1} \Rightarrow Y = \frac{(s+1) + 1 - e^{-2s-2}}{(s+1)(s+4)}$

$Y = \frac{1}{s+4} + \frac{1}{(s+1)(s+4)} - \frac{e^{-2s-2}}{(s+1)(s+4)} \qquad \left\{ \frac{1}{(s+1)(s+4)} = \frac{1}{3}\left[\frac{1}{s+1} - \frac{1}{s+4}\right] \right.$

$= \frac{2}{3} \frac{1}{s+4} + \frac{1}{3} \frac{1}{s+1} - \frac{1}{3e^2} e^{-2s} \left[\frac{1}{s+1} - \frac{1}{s+4}\right]$

$y = L^{-1}(Y) = \frac{2}{3} e^{-4t} + \frac{1}{3} e^{-t} - \frac{1}{3e^2} \left[u(t-2) e^{-(t-2)} - u(t-4) e^{-(t-4)}\right]$

NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER 1 EXAMINATION 2019-2020

MA2006 – ENGINEERING MATHEMATICS

November/December 2019 Time Allowed: 2½ hours

INSTRUCTIONS

1. This paper contains **FOUR (4)** questions and comprises **FIVE (5)** pages.
2. Answer **ALL** questions.
3. All questions carry equal marks.
4. This is a **CLOSED-BOOK** examination.
5. Mathematical tables and formulae are provided on pages 4 and 5.

1(a) For the two $n \times n$ matrices **A** and **B**, given:

$$\mathbf{A} + \mathbf{B} + \mathbf{AB} = \mathbf{0} \text{ (all the elements of the matrix } \mathbf{0} \text{ are 0)}$$

Prove: **AB** =**BA**.

(7 marks)

(b) Consider the following system of linear equations,

$$\begin{aligned} x_1 + 2x_2 + ax_3 &= 3 \\ ax_2 - 4x_3 &= 6 \\ 2x_1 + 5x_2 + ax_3 &= b \end{aligned}$$

where a and b are unknown real constants. Find the values of a and b such that the equations have

(i) unique solution,
(ii) no solution and
(iii) infinitely many solutions.

(9 marks)

(c) For the matrix

$$\mathbf{A} = \begin{bmatrix} 7 & 0 & -6 \\ 0 & -7 & -6 \\ -6 & -6 & a \end{bmatrix},$$

where a is a real number. It is known that matrix **A** has an eigenvalue 11. Find the value of a, all the eigenvalues and eigenvectors of matrix **A**.

(9 marks)

In Question 2, Cartesian coordinates are denoted by the usual symbols x, y and z.

2(a) It is given that ϕ and ψ are functions of x, y and z. If $\text{div}(\text{grad}(\phi)) = 0$ and $\text{div}(\text{grad}(\psi)) = 0$ at all points (x, y, z) in space, while S is any given enclosed surface, use Gauss theorem to prove that

$$\iint_S (\underline{\mathbf{n}} \bullet [\phi \text{ grad}(\psi)] - \underline{\mathbf{n}} \bullet [\psi \text{ grad}(\phi)])\, d\sigma = 0.$$

Gauss theorem: if T is the region enclosed by the closed surface S whose outward unit normal vector is $\underline{\mathbf{n}}$ and if $\underline{\mathbf{u}}$ is a vector function of x, y and z such that $\text{div}(\underline{\mathbf{u}})$ exists in T, then

$$\iint_S \underline{\mathbf{u}} \bullet \underline{\mathbf{n}}\, d\sigma = \iiint_T \text{div}(\underline{\mathbf{u}})\, dV.$$

(7 marks)

(b) Evaluate each of the following line integrals:

(i) $\int_C x\,ds$, where C is the portion of the curve $x = 2t$, $y = t^2$ and $z = -t$ between (2,1,1) and (4,4,-2).

(4 marks)

(ii) $\int_C \mathbf{F} \bullet d\mathbf{r}$, where $\mathbf{F} = xy\,\mathbf{i} + z\,\mathbf{j} + y^2\,\mathbf{k}$, $d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}$ and C is the directed straight line segment from (0, 0, 0) to (1, 1, 1).

(5 marks)

(c) Let S be the portion of the surface $z = 1 + x^2 + y^2$ for $z \le 10$. Find the area of the surface S.

(9 marks)

45

3(a) A periodic function is defined in one period as

$$f(x)=\begin{cases}1, & 0\le x<\frac{\pi}{2};\\ \frac{4}{\pi}(\pi-x), & \frac{\pi}{2}\le x\le\pi.\end{cases}$$

(i) Sketch $f(x)$ over the interval $-\pi\le x<\pi$. (3 marks)

(ii) Find the Fourier series for $f(x)$. (10 marks)

(iii) By using the Fourier series obtained in 3(a) (ii), find the value of following infinite series:

$$1+\frac{1}{9}+\frac{1}{25}+\frac{1}{49}+\cdots+\frac{1-(-1)^n}{2n^2},\ n=1,2,3,\cdots\to\infty \text{ are integers.}$$

(4 marks)

(b) If the function $f(x)$ in 3(a) is not periodic but is only defined in the interval $0\le x<\pi$, a Fourier sine series can be obtained for $f(x)$ by a proper expansion.

(i) Sketch the expansion for $f(x)$ over the interval $-\pi\le x\le 3\pi$. (4 marks)

(ii) Find the Fourier sine series for $f(x)$. (4 marks)

4(a) Find the Laplace transform to the function $f(t)$ defined as

$$f(t)=\begin{cases}0, & 0\le t<\pi/2;\\ t\sin(t), & \pi/2\le t<\pi;\\ 0, & t>\pi.\end{cases}$$

(9 marks)

(b) Find the inverse Laplace transform

$$L^{-1}\left\{\frac{s^2-s+20}{(s-6)(s^2-2s+26)}\right\}.$$

(6 marks)

(c) Solve the following ordinary differential equation for $y(t)$ by using the Laplace transform. (10 marks)

$$y''-3y'+2y=\begin{cases}0, & 0\le t\le 1;\\ 1, & 1<t\le 2;\\ 0, & t>2.\end{cases}\qquad y(0)=y'(0)=0.$$

FORMULAE FOR VECTOR CALCULUS PART

1. $\vec{a}=a_1\vec{i}+a_2\vec{j}+a_3\vec{k},\quad \vec{b}=b_1\vec{i}+b_2\vec{j}+b_3\vec{k},$

$\vec{a}\bullet\vec{b}=a_1b_1+a_2b_2+a_3b_3,$

$$\vec{a}\times\vec{b}=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\a_1&a_2&a_3\\b_1&b_2&b_3\end{vmatrix}=(a_2b_3-b_2a_3)\vec{i}+(a_3b_1-b_3a_1)\vec{j}+(a_1b_2-b_1a_2)\vec{k}.$$

2. $grad\ f=\nabla f=\frac{\partial f}{\partial x}\vec{i}+\frac{\partial f}{\partial y}\vec{j}+\frac{\partial f}{\partial z}\vec{k}$, for scalar function $f(x,y,z)$.

3. $\vec{V}=p(x,y,z)\vec{i}+q(x,y,z)\vec{j}+r(x,y,z)\vec{k},$

$\nabla\bullet\vec{V}=div\vec{V}=\frac{\partial p}{\partial x}+\frac{\partial q}{\partial y}+\frac{\partial r}{\partial z},$

$$\nabla\times\vec{V}=curl\,\vec{V}=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\\frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\p&q&r\end{vmatrix}=\left(\frac{\partial r}{\partial y}-\frac{\partial q}{\partial z}\right)\vec{i}+\left(\frac{\partial p}{\partial z}-\frac{\partial r}{\partial x}\right)\vec{j}+\left(\frac{\partial q}{\partial x}-\frac{\partial p}{\partial y}\right)\vec{k},$$

4. $ds=\sqrt{(dx)^2+(dy)^2+(dz)^2}=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt,$

$$\int_C f(x,y,z)ds=\int_{t_1}^{t_2}f[x(t),y(t),z(t)]\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt.$$

5. $\vec{r}=x\vec{i}+y\vec{j}+z\vec{k},\quad d\vec{r}=dx\vec{i}+dy\vec{j}+dz\vec{k}$, For $\vec{F}=p(x,y,z)\vec{i}+q(x,y,z)\vec{j}+r(x,y,z)\vec{k}$,

$$\int_C\vec{F}\bullet d\vec{r}=\int pdx+qdy+rdz=\int_{t_1}^{t_2}\left(p\frac{dx}{dt}+q\frac{dy}{dt}+r\frac{dz}{dt}\right)dt.$$

6. Green's Theorem

$$\oint_C f(x,y)dx+g(x,y)dy=\iint_R\left(\frac{\partial g}{\partial x}-\frac{\partial f}{\partial y}\right)dA$$

7. Surface integral:

$$\iint_S g(x,y,z)d\sigma=\iint_R g[x,y,z(x,y)]\sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}\,dA,$$ for the surface given by $z=f(x,y)$.

FORMULAE FOR SPECIAL FUNCTIONS

$\sin(x\pm y)=\sin x\cos y\pm\cos x\sin y\qquad \sin x\sin y=[-\cos(x+y)+\cos(x-y)]/2$

$\cos(x\pm y)=\cos x\cos y\mp\sin x\sin y\qquad \cos x\cos y=[\cos(x+y)+\cos(x-y)]/2$

$\sin x\cos y=[\sin(x+y)+\sin(x-y)]/2$

$\sin^2 x=(1-\cos 2x)/2,\quad \cos^2 x=(1+\cos 2x)/2$

$\sin x+\sin y=2\sin\frac{x+y}{2}\cos\frac{x-y}{2},\quad \sin x-\sin y=2\cos\frac{x+y}{2}\sin\frac{x-y}{2}$

$\cos x+\cos y=2\cos\frac{x+y}{2}\cos\frac{x-y}{2},\quad \cos x-\cos y=-2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$

$\sinh x=(e^x-e^{-x})/2,\quad \cosh x=(e^x+e^{-x})/2,\ \sinh x=-i\sin ix,\quad \cosh x=\cos ix$

1.a) $A+B+AB=0 \Rightarrow B=-A(I+B)$, $A=-B(I+A)$

$\therefore AB=-B(I+A)(-A)(I+B)$
$=(-B-BA)(-A)(I+B)$
$=BA+BAA+BAB+BAAB$

$BA=-A(I+B)(-B)(I+A)$
$=(-A-AB)(-B)(I+A)$
$=AB+ABB+ABA+ABBA.$

If $AB=BA$,
then $AB=AB+ABA+ABB+ABBA=BA$ (Proven)

*Go ask Prof, this might be wrong.

1.b.i) Let $A=\begin{bmatrix}1&2&a\\0&a&-4\\2&5&a\end{bmatrix}$

$\det(A)=a^2-16-2a^2+20=4-a^2$

For $\det(A)=0$, $a^2=4$, $\therefore a=\pm2$.

For unique solution, $\det(A)\neq0$. $\therefore a\neq2$ and $a\neq-2$. $b\in\mathbb{R}$.

1.b.ii) $\left[\begin{array}{ccc|c}1&2&a&3\\0&a&-4&6\\2&5&a&b\end{array}\right] \rightarrow \left[\begin{array}{ccc|c}1&2&a&3\\0&a&-4&6\\0&0&-a^2+4&a(b-6)-6\end{array}\right]$

For no solution, $a(b-6)-6\neq0$.

For $a(b-6)-6=0$, $\Rightarrow b-6=\frac{6}{a} \Rightarrow b=\frac{6}{a}+6$.

$\therefore$ For no solution: $a=2$ and $b\neq9$ OR $a=-2$ and $b\neq3$.

1.b.iii) For infinitely many solutions, $a=2$ and $b=9$ OR $a=-2$ and $b=3$

1.c) $A-\lambda I=\begin{bmatrix}7-\lambda&0&-6\\0&-7-\lambda&-6\\-6&-6&a-\lambda\end{bmatrix}$

$\det(A-\lambda I)=(7-\lambda)(-7-\lambda)(a-\lambda)-(-6)(-7-\lambda)(-6)$
$-(-6)(-6)(7-\lambda)$
$=(-49+\lambda^2)(a-\lambda)-36(-7-\lambda)-36(7-\lambda)$
$=-49a+49\lambda+a\lambda^2-\lambda^3+252+36\lambda-252+36\lambda$
$=-\lambda^3+a\lambda^2+121\lambda-49a$

When $\lambda=11$, $-1331+121a+1331-49a=0$ $\therefore a=0$

$\therefore -\lambda^3+121\lambda=0$
$\lambda(-\lambda^2+121)=0$
$\lambda(-\lambda+11)(\lambda+11)=0$
$\therefore \lambda_1=0$ OR $\lambda_2=11$ OR $\lambda_3=-11$.

For $\lambda_1 = 0$, $\left(\begin{array}{ccc|c} 7 & 0 & -6 & 0 \\ 0 & -7 & -6 & 0 \\ -6 & -6 & 0 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 7 & 0 & -6 & 0 \\ -7 & -7 & 0 & 0 \\ -6 & -6 & 0 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 7 & 0 & -6 & 0 \\ -7 & -7 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$\therefore x = t,\ y = -t,\ z = 7/6\, t$ Eigenvector is : $\frac{t}{6}\begin{pmatrix} 6 \\ -6 \\ 7 \end{pmatrix}$

For $\lambda_2 = 11$, $\left(\begin{array}{ccc|c} -4 & 0 & -6 & 0 \\ 0 & -18 & -6 & 0 \\ -6 & -6 & -11 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 0 & 12 & 4 & 0 \\ 0 & -18 & -6 & 0 \\ -6 & -6 & -11 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & -18 & -6 & 0 \\ -6 & -6 & -11 & 0 \end{array}\right)$

$\therefore z = t,\ y = -\frac{1}{3}t,\ x = \frac{-3}{2}t$ Eigenvector is : $\frac{t}{6}\begin{pmatrix} -9 \\ -2 \\ 6 \end{pmatrix}$

For $\lambda_3 = -11$, $\left(\begin{array}{ccc|c} 18 & 0 & -6 & 0 \\ 0 & 4 & -6 & 0 \\ -6 & -6 & 11 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 0 & -18 & 27 & 0 \\ 0 & 4 & -6 & 0 \\ -6 & -6 & 11 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 4 & -6 & 0 \\ -6 & -6 & 11 & 0 \end{array}\right)$

$\therefore z = t,\ y = \frac{6}{4}t,\ x = \frac{1}{3}t$ Eigenvector is : $\frac{t}{6}\begin{pmatrix} 2 \\ 9 \\ 6 \end{pmatrix}$

2.a) $\iint_S \left(\underline{n}\cdot[\phi\,\text{grad}(\psi)] - \underline{n}\cdot[\psi\,\text{grad}(\phi)]\right) d\sigma = \iiint_T \left(\text{div}[\phi\,\text{grad}(\psi)] - \text{div}[\psi\,\text{grad}(\phi)]\right) dV$

$= \iiint_T \{\phi\,\text{div}(\text{grad}(\psi)) + \text{grad}(\phi)\cdot\text{grad}(\psi) - \psi\,\text{div}(\text{grad}(\phi)) - \text{grad}(\psi)\,\text{grad}(\phi)\}\,dV$

$= \iiint_T [\,0 + \nabla\phi\cdot\nabla\psi - 0 - \nabla\psi\cdot\nabla\phi\,]\,dV$

$= \iiint_T 0\,dV$

$= 0$ (Proven)

2.b.i) At $(2,1,-1)$, $t=1$. At $(4,4,-2)$, $t=2$

$ds = \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right]^{1/2} dt.$

$= \left[(2)^2 + (2t)^2 + (-1)^2\right]^{1/2} dt$

$= \sqrt{5+4t^2}\,dt.$

$\therefore \int_C x\,ds = \int_1^2 2t\sqrt{5+4t^2}\,dt. = \left[\frac{1}{6}(5+4t^2)^{3/2}\right]_1^2 = \frac{1}{6}(21)^{3/2} - \frac{1}{6}(9)^{3/2}$

$= \frac{21}{6}\sqrt{21} - \frac{27}{6}$

2.b.ii) Eqn of line: $x = y = z = t.$

$\int_C \underline{F}\cdot d\underline{r} = \int_C xy\,dx + z\,dy + y^2\,dz = \int_0^1 t^2(1) + t(1) + t^2(1)\,dt$

$= \int_0^1 2t^2 + t\,dt$

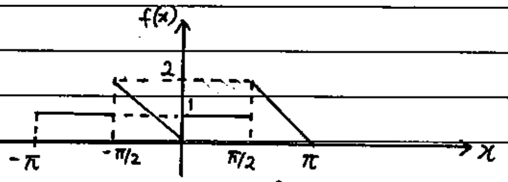
$= \left[\frac{2}{3}t^3 + \frac{1}{2}t^2\right]_0^1$

$= 2/3 + \frac{1}{2} = 7/6$

2.c) ~~$\text{Area} = \iint_S dA = \int_0^{2\pi}\int_0^3 r\,dr\,d\theta = \int_0^{2\pi}\left[\frac{1}{2}r^2\right]_0^3 d\theta = \int_0^{2\pi}\frac{9}{2}\,d\theta = \left[\frac{9}{2}\theta\right]_0^{2\pi} = 9\pi$~~

$$\text{Area} = \iint_S d\sigma$$
$$= \iint_R \left[1+(2x)^2+(2y)^2\right]^{1/2} dA$$
$$= \int_0^{2\pi}\int_0^3 \sqrt{1+(2r\cos\theta)^2+(2r\sin\theta)^2}\cdot r\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^3 \sqrt{1+4r^2}\cdot r\,dr\,d\theta$$
$$= \int_0^{2\pi}\left[\frac{1}{12}(1+4r^2)^{3/2}\right]_0^3 d\theta$$
$$= \int_0^{2\pi}\frac{1}{12}(37)^{3/2}-\frac{1}{12}\,d\theta$$
$$= 2\pi\left[\frac{1}{12}(37)^{3/2}-\frac{1}{12}\right]$$
$$= \frac{\pi}{6}(37)^{3/2}-\frac{\pi}{6}$$

3.a.i)



3.a.ii) $P = 2L = \pi \Rightarrow L = \pi/2$.

$$a_0 = \frac{1}{\pi}\int_0^{\pi/2} 1\,dx + \frac{1}{\pi}\int_{\pi/2}^{\pi}\frac{4}{\pi}(\pi - x)\,dx$$
$$= \frac{1}{\pi}\left\{\frac{\pi}{2} + \int_{\pi/2}^{\pi} 4 - \frac{4}{\pi}x\,dx\right\}$$
$$= \frac{1}{\pi}\left\{\frac{\pi}{2} + \left[4x - \frac{2}{\pi}x^2\right]_{\pi/2}^{\pi}\right\}$$
$$= \frac{1}{\pi}\left\{\frac{\pi}{2} + 4\pi - 2\pi - 2\pi + \frac{\pi}{2}\right\} = 1$$

* $\int x\cos 2nx\,dx = x\frac{1}{2n}\sin 2nx - \int\frac{1}{2n}\sin 2nx\,dx = \frac{x}{2n}\sin(2nx) + \frac{1}{4n^2}\cos(2nx) + C$

* $\int x\sin 2nx\,dx = x\frac{(-1)}{2n}\cos 2nx + \int\frac{1}{2n}\cos 2nx\,dx = \frac{-x}{2n}\cos(2nx) + \frac{1}{4n^2}\sin(2nx) + C$

$$a_n = \frac{2}{\pi}\left\{\int_0^{\pi/2}\cos 2nx\,dx + \int_{\pi/2}^{\pi}\left(4 - \frac{4}{\pi}x\right)\cos 2nx\,dx\right\}$$
$$= \frac{2}{\pi}\left\{\left[\frac{1}{2n}\sin 2nx\right]_0^{\pi/2} + \int_{\pi/2}^{\pi}4\cos 2nx\,dx - \frac{4}{\pi}\int_{\pi/2}^{\pi}x\cos 2nx\,dx\right\}$$
$$= \frac{2}{\pi}\left\{0 + \left[\frac{2}{n}\sin 2nx\right]_{\pi/2}^{\pi} - \frac{4}{\pi}\left[\frac{x}{2n}\sin(2nx) + \frac{1}{4n^2}\cos(2nx)\right]_{\pi/2}^{\pi}\right\}$$
$$= \frac{2}{\pi}\left\{0 + 0 - \frac{4}{\pi}\left[\frac{1}{4n^2} - \frac{1}{4n^2}(-1)^n\right]\right\}$$
$$= \frac{2}{\pi^2 n^2}\left[(-1)^n - 1\right]$$

$$b_n = \frac{2}{\pi}\left\{\int_0^{\pi/2}\sin 2nx\,dx + \int_{\pi/2}^{\pi}4\sin 2nx\,dx - \frac{4}{\pi}\int_{\pi/2}^{\pi}x\sin 2nx\,dx\right\}$$
$$= \frac{2}{\pi}\left\{\left[\frac{-1}{2n}\cos 2nx\right]_0^{\pi/2} + \left[\frac{-2}{n}\cos 2nx\right]_{\pi/2}^{\pi} - \frac{4}{\pi}\left[\frac{-x}{2n}\cos(2nx) + \frac{1}{4n^2}\sin(2nx)\right]_{\pi/2}^{\pi}\right\}$$
$$= \frac{2}{\pi}\left\{\frac{-1}{2n}(-1)^n + \frac{1}{2n} - \frac{2}{n} + \frac{2}{n}(-1)^n - \frac{4}{\pi}\left[\frac{-\pi}{2n} + \frac{\pi}{4n}(-1)^n\right]\right\}$$
$$= \frac{2}{\pi}\left\{(-1)^n\left(\frac{3}{2n} - \frac{1}{n}\right) + \frac{1}{2n} - \frac{2}{n} + \frac{2}{n}\right\}$$
$$= \frac{2}{\pi}\left\{\frac{(-1)^n}{2n} + \frac{1}{2n}\right\}$$
$$= \frac{1}{n\pi}\left[(-1)^n + 1\right]$$

$$\therefore f(x) = 1 + \sum_{n=1}^{\infty}\left\{\frac{2}{\pi^2 n^2}\left[(-1)^n - 1\right]\cos(2nx) + \frac{1}{n\pi}\left[(-1)^n + 1\right]\sin(2nx)\right\}$$

3.a.iii) By looking at the fourier series, we can see that when n is odd, we only look at the cos term, but when n is even, we only look at the sin term. Looking at the infinite series, we want $\frac{1}{n^2}$ term, thus we only want the cos term. To make this possible, assume $x = \pi/2$

$\sum_{n=1}^{\infty} \left\{ \frac{2}{\pi^2 n^2}[(-1)^n - 1]\cos(2nx) + \frac{1}{\pi n}[(-1)^n + 1]\sin(2nx) \right\}$ ~~= A lets assume this~~

~~When n~~ First term $(n=1)$ $= \frac{-4}{\pi^2}$

Third term $(n=3)$ $= \frac{-4}{9\pi^2}$

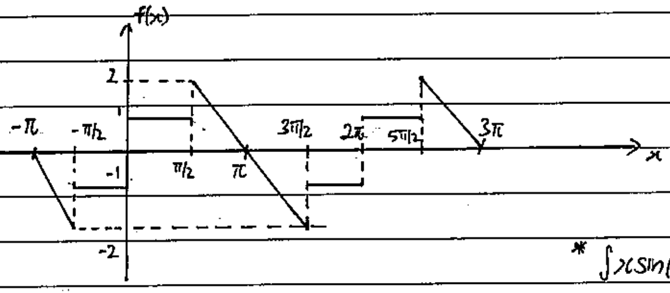
Fifth term $(n=5)$ $= \frac{-4}{25\pi^2}$

$\therefore$ Value of infinite series $= f(\pi/2)\left(\frac{-\pi^2}{4}\right) - 1$

$= \frac{3}{2}\left(\frac{-\pi^2}{4}\right) - 1$

$= -\frac{3}{8}\pi^2 - 1$

3.b.i)



* $\int x\sin(nx)\,dx = \frac{-x}{n}\cos(nx) + \int \frac{1}{n}\cos(nx)\,dx$

$= \frac{-x}{n}\cos(nx) + \frac{1}{n^2}\sin(nx) + C$

3.b.ii) $P = 2L = 2\pi \Rightarrow L = \pi$

$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$

$= \frac{2}{\pi}\left\{ \int_0^{\pi/2} \sin(nx)\,dx + \int_{\pi/2}^{\pi} 4\sin(nx)\,dx - \frac{4}{\pi}\int_{\pi/2}^{\pi} x\sin(nx)\,dx \right\}$

$= \frac{2}{\pi}\left\{ \left[\frac{-1}{n}\cos(nx)\right]_0^{\pi/2} + \left[\frac{-4}{n}\cos(nx)\right]_{\pi/2}^{\pi} - \frac{4}{\pi}\left[\frac{-x}{n}\cos(nx) + \frac{1}{n^2}\sin(nx)\right]_{\pi/2}^{\pi} \right\}$

$= \frac{2}{\pi}\left\{ \frac{-1}{n}\cos\left(\frac{n\pi}{2}\right) + \frac{1}{n} - \frac{4}{n}(-1)^n + \frac{4}{n}\cos\left(\frac{n\pi}{2}\right) - \frac{4}{\pi}\left[\frac{-\pi}{n}(-1)^n + \frac{\pi}{2n}\cos\left(\frac{n\pi}{2}\right) - \frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\right] \right\}$

$= \frac{2}{\pi}\left\{ \frac{3}{n}\cos\left(\frac{n\pi}{2}\right) + \frac{1}{n} - \frac{4}{n}(-1)^n + \frac{4}{n}(-1)^n - \frac{2}{n}\cos\left(\frac{n\pi}{2}\right) + \frac{4}{\pi n^2}\sin\left(\frac{n\pi}{2}\right) \right\}$

$= \frac{2}{\pi}\left\{ \frac{1}{n}\cos\left(\frac{n\pi}{2}\right) + \frac{1}{n} + \frac{4}{\pi n^2}\sin\left(\frac{n\pi}{2}\right) \right\}$

$= \frac{2}{\pi n}\cos\left(\frac{n\pi}{2}\right) + \frac{2}{n\pi} + \frac{8}{\pi^2 n^2}\sin\left(\frac{n\pi}{2}\right)$

$\therefore f(x) = \sum_{n=1}^{\infty} \left\{ \left[\frac{2}{\pi n}\cos\left(\frac{n\pi}{2}\right) + \frac{2}{n\pi} + \frac{8}{\pi^2 n^2}\sin\left(\frac{n\pi}{2}\right)\right]\sin(nx) \right\}$

4.c) $y'' - 3y' + 2y = u(t-1)(1) - u(t-2)(1)$

Taking laplace transform,

$$s^2Y - sy(0) - y'(0) - 3(sY - y(0)) + 2Y = e^{-s}\left(\frac{1}{s}\right) - e^{-2s}\left(\frac{1}{s}\right)$$

$$(s^2 - 3s + 2)Y = \frac{1}{s}(e^{-s} - e^{-2s})$$

$$Y = \frac{1}{s(s^2-3s+2)}(e^{-s} - e^{-2s})$$

$$= \frac{1}{s(s-2)(s-1)}(e^{-s} - e^{-2s})$$

$$\frac{1}{s(s-2)(s-1)} = \frac{A}{s} + \frac{B}{s-2} + \frac{C}{s-1}$$

$$1 = A(s^2-3s+2) + B(s^2-s) + C(s^2-2s)$$

$A + B + C = 0$ → $B + C = -\frac{1}{2}$ — ①

$-3A - B - 2C = 0$ → $B + 2C = -3/2$ — ②

$2A = 1 \Rightarrow A = 1/2$

② − ①, $C = -1$

$\therefore B = 1/2$

$$\therefore Y(s) = (e^{-s} - e^{-2s})\left(\frac{1/2}{s} + \frac{1/2}{s-2} - \frac{1}{s-1}\right)$$

$$y(t) = u(t-1)\left[\frac{1}{2} + \frac{1}{2}e^{2(t-1)} - e^{t-1}\right] - u(t-2)\left[\frac{1}{2} + \frac{1}{2}e^{2(t-2)} - e^{t-2}\right]$$

DISCLAIMER: *The solutions are done by students who scored A or above in this subject. MAE Club and Campus supplies are not liable or responsible for any errors in the contents of these solutions. Students are advised to take the solutions as a guide rather than absolute answers to exam paper.*

Should there be any mistake identified, please proceed to the Facebook link encoded in the QR code to feedback or submit correct answers. The link is: http://bit.ly/2lW2C32

4.a) $f(t) = u(t-\frac{\pi}{2})\, t\sin t - u(t-\pi)\, t\sin(t)$

$= u(t-\frac{\pi}{2})(t-\frac{\pi}{2})\sin t + u(t-\frac{\pi}{2})(\frac{\pi}{2})\sin t - u(t-\pi)(t-\pi)\sin t - u(t-\pi)(\pi)\sin t$

$= u(t-\frac{\pi}{2})(t-\frac{\pi}{2})\cos(t-\pi/2) + u(t-\pi/2)(\frac{\pi}{2})\cos(t-\pi/2) + u(t-\pi)(t-\pi)\sin(t-\pi)$

$+ u(t-\pi)(\pi)\sin(t-\pi)$

$\mathcal{L}[\cos t] = \frac{s}{s^2+1}$, $\mathcal{L}(t\cos t) = -\frac{d}{ds}\left(\frac{s}{s^2+1}\right) = -\left[\frac{1}{s^2+1} + s(-1)(s^2+1)^{-2}(2s)\right]$

$= -\left[\frac{1}{s^2+1} - \frac{2s^2}{(s^2+1)^2}\right]$

$= \frac{-s^2-1+2s^2}{(s^2+1)^2} = \frac{s^2-1}{(s^2+1)^2}$

$\mathcal{L}(\sin t) = \frac{1}{s^2+1}$, $\mathcal{L}(t\sin t) = -\frac{d}{ds}\left(\frac{1}{s^2+1}\right)$

$= -\left[(-1)(s^2+1)^{-2}(2s)\right]$

$= \frac{2s}{(s^2+1)^2}$

$\mathcal{L}(f(t)) = e^{-\frac{\pi}{2}s}\left(\frac{s^2-1}{(s^2+1)^2}\right) + e^{-\frac{\pi}{2}s}\left(\frac{\pi}{2}\right)\left(\frac{s}{s^2+1}\right) + e^{-\pi s}\left(\frac{2s}{(s^2+1)^2}\right) + e^{-\pi s}(\pi)\left(\frac{1}{s^2+1}\right)$

4.b) $\frac{s^2-s+20}{(s-6)(s^2-2s+26)} = \frac{s^2-s+20}{(s-6)[(s-1)^2+25]} = \frac{As+B}{(s-1)^2+25} + \frac{C}{s-6}$

$s^2-s+20 = (As+B)(s-6) + C(s^2-2s+26)$

$= As(s-6) + B(s-6) + C(s^2-2s+26)$

$\therefore 1 = A+C$ — ① From ①, $A = 1-C$ — ④

$-1 = -6A + B - 2C$ — ② Sub. ④ into ②, $-1 = -6(1-C)+B-2C$

$20 = -6B + 26C$ — ③ $5 = 4C + B$ —

$B = 5-4C$ — ⑤

Sub. ⑤ into ③, $20 = -6(5-4C) + 26C$

$20 = -30 + 24C + 26C$

$\therefore C = 1,\ A = 0,\ B = 1$

$\therefore \mathcal{L}^{-1}\left\{\frac{s^2-s+20}{(s-6)(s^2-2s+26)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2+25} + \frac{1}{s-6}\right\}$

$= e^{t}\frac{\sin 5t}{5} + e^{6t}$