NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER 2 EXAMNINATION 2011-2012

AE4613 – AEROELASTICITY

April/May 2012 — Time Allowed: 2 ½ hours

INSTRUCTIONS

1. This paper contains **FOUR** (4) questions and has **FOUR** (4) pages.
2. Answer **ALL** questions.
3. All questions carry equal marks.
4. This is an **Open-Book Examination.**

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1. A simply-supported uniform beam of length $l=1m$, having a rectangular cross-section of width $b=10cm$ and thickness $h=5cm$ is shown in Figure 1. Young's modulus E is assumed to be $E=2.1\times10^{11}\ N/m^2$ and density $\rho A=40kg/m$.

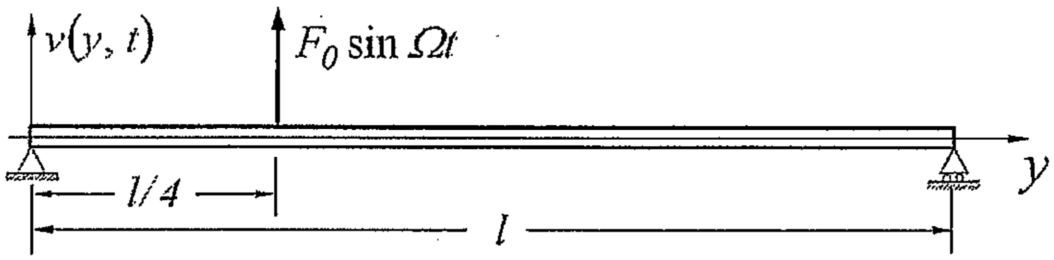


Figure 1 – A simply-supported rectangular beam

(a) In the case of free vibration ($F_0=0$), determine the bending vibration response of the beam due to following given initial displacement and velocity conditions:

$$v(y,\ 0) = 0.1\sin\pi y + 0.2\sin 2\pi y,$$
$$\dot{v}(y,\ 0) = 0.$$

(12 marks)

(b) Under the same given initial conditions, if an external force $F=100\sin100t$ is applied at $y=l/4$ as shown in Figure 1, determine the subsequent bending vibration of the beam.

(13 marks)

2. A rigid model wing with aileron is elastically supported in a wind tunnel test, as shown in Figure 2. The torsional stiffness k_T is provided through two uniform circular mild steel beams of length $l=0.5m$, diameter $d=0.02m$ and shear modulus $G=9\times10^{11}$ N/m^2. Geometric parameters of the wing are: $S=0.8m^2$, $c=0.4m$, $e=0.1m$, $a=0.12m$. Aerodynamic coefficients are: $C_{L\alpha}=6\ rad^{-1}$, $C_{M0}=0.0\ rad^{-1}$, $C_{L\beta}=1.2\ rad^{-1}$ and $C_{M\beta}=-0.35\ rad^{-1}$. Air density is assumed to be $\rho=1.223\ kg/m^3$.

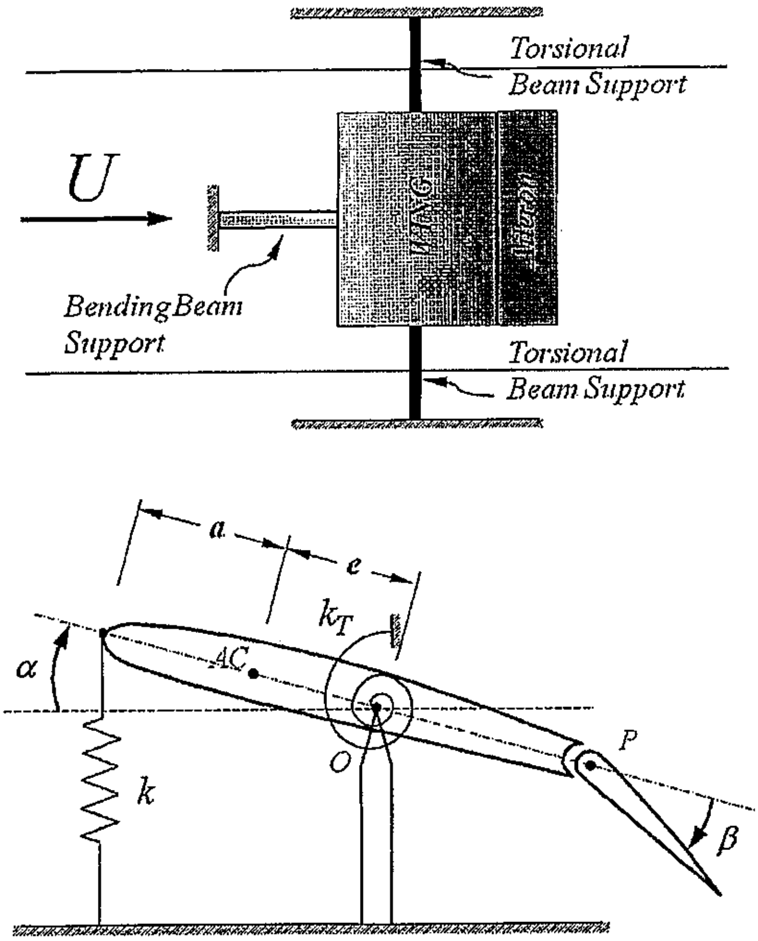


Figure 2: An elastically restrained wing and its structural model

(a) For the case of no leading edge support ($k = 0$), determine the divergence speed U_D and the reversal speed U_R of the system;

(8 marks)

(b) In order to enhance further the divergence speed of the system by *50%*, a linear spring k is added to the leading edge by attaching a cantilevered bending beam as shown in Figure 2. Determine the value of the spring constant k required and the length of the uniform cantilevered beam if its cross-sectional rigidity $EI=10,000\ Nm^2$;

(10 marks)

Note: Question No. 2 continues on page 3.

(c) For the enhanced system, determine the aileron efficiency η when airspeed U becomes $U=200\ m/s$ and explain the practical implications of the efficiency value you have obtained.

(7 marks)

Hint: The torsional stiffness k_T of a uniform beam with one end fixed and the bending stiffness k of a cantilevered uniform beam can be obtained as,

$$k_T = \frac{GI_P}{l}, \qquad k = \frac{3EI}{l^3}.$$

3. A rigid flat wing section with a chord length $c=2m$ is pivoted at A with $a=0.4m$ in a wind tunnel test as shown in Figure 3. The wing is restrained by an elastic spring with a spring constant K and the free stream air speed is assumed to be $U=125m/s$. If the wing is given a known pitching vibration (through some built-in actuation mechanism) of $\alpha(t) = 0.02\sin 15t$, determine:

(a) the Theodorsen Function $C(k)$ and its magnitude and phase corresponding to the given pitching vibration;

(6 marks)

(b) the circulatory aerodynamic lift L_c generated due to the given motion of the wing;

(10 marks)

(c) if the given pitching vibration is changed to become $\alpha(t) = 0.02\sin 150t$, derive the new circulatory aerodynamic lift L_c and comment on its change due to increased frequency of oscillation.

(9 marks)

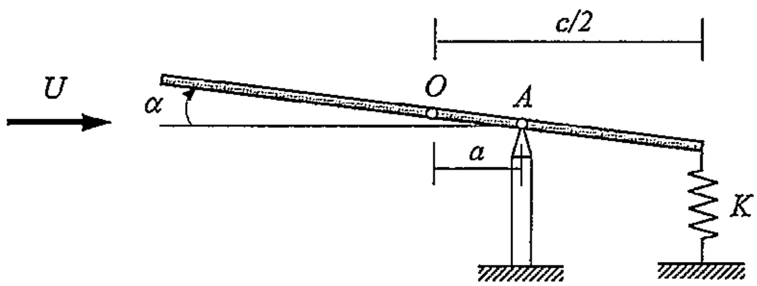


Figure 3: A flat wing undergoing wind tunnel test

4(a) A wing bending-torsion system is modeled in terms of coordinates α and θ:

$$\begin{bmatrix} 12D^2 + 6VD + (4\times 10^5 - 9V^2) & 3VD + 3V^2 \\ -3V^2 & D^2 + VD + V^2 \end{bmatrix} \begin{bmatrix} \alpha \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

where $D=d/dt$. Determine the critical flutter speeds and their corresponding frequencies.

(10 marks)

(b) The aircraft panel shown in Figure 4 below is subjected to a supersonic flow of speed V along its surface. The panel consists of three rigid plates of length L and Mass M or $2M$, hinged at each end, and attached to springs of stiffness K and $2K$, as shown in the figure.

(i) Making use of piston theory, Determine X_F and λ_F, when panel flutter occurs. They are respectively defined as

$$X^2 = \frac{\mu_1 L\omega^2}{k_1} \quad \text{and} \quad \lambda = \frac{\rho V^2}{2Mk_1}$$

(ii) At natural frequency, (i.e. $\lambda=0$), determine X_N. By comparing X_F and X_N, discuss the mode-shapes.

Hint: you can assume the generalized coordinates, $q_1 = \hat{q}_1 e^{j\omega t}$ **and** $q_2 = \hat{q}_2 e^{j\omega t}$

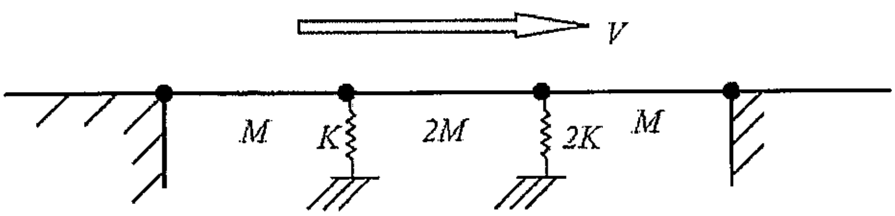


Figure 4 - Aircraft panel subjected to a supersonic flow with a speed *V*

(15 marks)

End of Paper

MA4704 Sem 2 11/12

1. (a) For a pinned-pinned boundary conditions, $\phi_i(y) = \sin(\alpha_i y) = \sin\left(\frac{i\pi y}{\ell}\right) = \sin(i\pi y)$

$v(y,t) = \sum_{i=1}^{\infty} \phi_i(y)\,(A_i \sin w_i t + B_i \cos w_i t)$

$\dot{v}(y,t) = \sum_{i=1}^{\infty} \phi_i(y)\,(A_i w_i \cos w_i t - B_i w_i \sin w_i t)$ $\qquad w_0 = (\alpha_i \ell)^2 \sqrt{\frac{EI}{\rho A \ell^4}} = 74\,(i\pi)^2$

$v(y,0) = \sum_{i=1}^{\infty} \sin(i\pi y) B_i = 0.1 \sin \pi y + 0.2 \sin 2\pi y$

$B_1 \sin(\pi y) + B_2 \sin 2\pi y = 0.1 \sin \pi y + 0.2 \sin 2\pi y$

$\therefore B_1 = 0.1$, $B_2 = 0.2$, B_3 and above $= 0$.

$\dot{v}(y,0) = \sum_{i=1}^{\infty} \phi_i(y) A_i w_i = 0 \Rightarrow A_i = 0$

$v(y,t) = \sum_{i=1}^{\infty} \phi_i(y)\,(B_i \cos w_i t)$

$= [\sin(\pi y)][0.1 \cos 74\pi^2 t] + [\sin(2\pi y)][0.2 \cos 296\pi^2 t]$

(b) $\theta(t) = \int_0^{\ell} F(y,t)\phi_i(y)\,dy = \int_0^1 100 \sin 100t\, \delta(y - 1/4) \sin(i\pi y)\,dy$

$= 100 \sin 100t \sin\left(\frac{i\pi}{4}\right)$

$\rho A \sigma_i \ddot{\xi}_i + \rho A \sigma_i w_i^2 \xi_i = \theta(t)$ $\qquad \sigma_i = \int_0^1 \sin(i\pi y)\sin(j\pi y) = \frac{\ell}{2} = \frac{1}{2}$

$\ddot{\xi}_i + w_i^2 \xi_i = \frac{1}{\rho A \sigma_i}\,\theta(t)$ $\qquad w_i = (\alpha_i \ell)^2 \sqrt{\frac{EI}{\rho A \ell^4}} = 74\,(i\pi)^2$

$\ddot{\xi}_i + [74 i^2 \pi^2]^2 \xi_i = 0.05\,[100 \sin 100t \sin(\frac{i\pi}{4})]$

$\ddot{\xi}_i + 5468.75 i^4 \pi^4 \xi_i = 5 \sin 100t \sin(\frac{i\pi}{4})$

Homogeneous Solution: $\xi_i = A_i \sin w_i t + B_i \cos w_i t$

Particular Solution: $\xi_i = C \cos 100t + D \sin 100t$

$\ddot{\xi}_i = -(100)^2 C \cos 100t - (100)^2 D \sin 100t$

$-(100)^2 C \cos 100t - (100)^2 D \sin 100t + 5468.75 i^4 \pi^4 (C \cos 100t + D \sin 100t) = 5 \sin 100t \sin(\frac{i\pi}{4})$

Comparing $\cos 100t$ terms, $-(100)^2 C + 5468.75 i^4 \pi^4 C = 0 \Rightarrow C = 0$

Comparing $\sin 100t$ terms, $-(100)^2 D + 5468.75 i^4 \pi^4 D = 5 \sin(\frac{i\pi}{4})$

$$D = \frac{5 \sin\left(\frac{i\pi}{4}\right)}{5468.75 i^4 \pi^4 - 100^2}$$

$\therefore \xi_i(t) = A_i \sin w_i t + B_i \cos w_i t + \frac{5 \sin(i\pi/4)}{5468.75 i^4 \pi^4 - 100^2} \sin 100t$

$v(y,t) = \sum_{i=1}^{\infty} \sin(i\pi y)\left[A_i \sin w_i t + B_i \cos w_i t + \frac{5 \sin(i\pi/4)}{5468.75 i^4 \pi^4 - 100^2} \sin 100t\right]$

$\dot{v}(y,t) = \sum_{i=1}^{\infty} \sin(i\pi y)\left[A_i w_i \cos w_i t - B_i w_i \sin w_i t + \frac{500 \sin(i\pi/4)}{5468.75 i^4 \pi^4 - 100^2} \cos 100t\right]$

$\dot{v}(y,0) = \sum_{i=1}^{\infty} \sin(i\pi y)\left(A_i w_i + \frac{500 \sin(i\pi/4)}{5468.75 i^4 \pi^4 - 100^2}\right) = 0$

Multiply both sides by $\sin(j\pi y)$ and integrate over $[0,1]$,

$\frac{\ell}{2}\left[A_i w_i + \frac{500 \sin(i\pi/4)}{5468.75 i^4 \pi^4 - 100^2}\right] = 0$

$A_i = -\frac{500 \sin(i\pi/4)}{w_i (5468.75 i^4 \pi^4 - 100^2)}$

$v(y,0) = \sum_{i=1}^{\infty} \sin(i\pi y) B_i = 0.1 \sin \pi y + 0.2 \sin 2\pi y$

After similar integration process in (a), we will obtain $B_i = 0$

$\therefore v(y,t) = \sum_{i=1}^{\infty} \sin(i\pi y)\left[-\frac{500 \sin(i\pi/4)}{74(i\pi)^2 (5468.75 i^4 \pi^4 - 100^2)} \sin 74 (i\pi)^2 t + \frac{5 \sin(i\pi/4)}{5468.75 i^4 \pi^4 - 100^2} \sin 100t\right]$

2. (a) $k = \frac{G I_p}{\ell} = \frac{(9\times10^{11})(5\times10^{-9}\pi)}{0.5}$ $\qquad I_p = \frac{\pi d^4}{32} = (5\times10^{-9})\pi$

$= 9000\pi\ Nm^{-1}$

Since there are 2 beams, $k_{T,total} = k \times 2 = 18000\pi\ Nm^{-1}$

$q_D = \frac{k_T}{S C_{l\alpha} e} = \frac{18000\pi}{0.8(6)(0.1)} = 37500\pi\ N/m^2$

$u_D = \sqrt{\frac{2 q_D}{\rho}} = \sqrt{\frac{2(37500\pi)}{1.223}} = 438.9\ ms^{-1}$

$q_R = -\frac{k_T C_{L\beta}}{C_{l\alpha} S c C_{m\beta}} = -\frac{18000\pi(1.2)}{6(0.8)(0.4)(-0.35)} = \frac{225000}{7}\pi\ N/m^2$

$u_R = \sqrt{\frac{2 q_R}{\rho}} = 406.4\ ms^{-1}$

(b) $u_{D,new} = 1.5 u_D = 658.3905649 \ ms^{-1}$

$q_{D,new} = \frac{1}{2}\rho u_{D,new}^2 = 265071.8801 \ N/m^2$

$\bar{k} = q_{D,new} S C_{L\alpha} e = 127234.5025 \ Nm^{-1}$

Moment equilibrium about Elastic Axis:

$$M_{ac} + Le = k_T\alpha + k\alpha(a+e)^2$$
$$= [k_T + (a+e)^2 k]\alpha = \bar{k}\alpha$$

$k_T + (a+e)^2 k = \bar{k}$

$18000\pi + (0.12+0.1)^2 k = 127234.5025$

$$k = 1460451.13 \ Nm^{-1} = \frac{3EI}{l^3}$$
$$l = 0.274m$$

(c) $q_{R,new} = -\frac{\bar{k} C_{L\beta}}{C_{L\alpha} S c C_{m\beta}} = -\frac{127234.5025(1.2)}{6(0.8)(0.4)(-0.35)} = 227204.4688 \ N/m^2$

$$\eta = \frac{1 - q/q_R}{1 - q/q_D} = \frac{1 - \frac{\frac{1}{2}(1.225)(200)^2}{227204.4688}}{1 - \frac{\frac{1}{2}(1.225)(200)^2}{265071.8801}} = ~~0.983 = 98.3\%~~$$

Since η is very close to 1, it implies that the enhanced system is behaving close to a rigid wing. The aileron efficiency η shows the change in lift per unit change in β for elastic wing work approximately to rigid wing for low speed. This means that an increase in β will increase L for small q (aerodynamic force is dominant).

3. (a) $\omega = 15$ rad/s $\qquad k = \frac{\omega b}{u} = \frac{\omega c}{2u} = \frac{15(2)}{2(125)} = 0.12$

$$C(k) = 1 - \frac{0.165}{1-(0.0455/k)i} - \frac{0.335}{1-(0.30/k)i}$$
$$= 1 - \frac{0.165}{1-(0.0455/0.12)i} - \frac{0.335}{1-(0.30/0.12)i}$$
$$= 1 - \frac{0.165(1+91/240 i)}{1+(91/240)^2} - \frac{0.335(1+2.5i)}{1+2.5^2}$$
$$= 0.809532998 - 0.1702158 64 i$$

$|C(k)| = 0.827234619 = 0.827 \qquad \text{Arg}\, C(k) = -\tan^{-1}\frac{12.17}{0.81} = -11.87428031 = -11.9°$

(b) $L_c = 2\pi\rho b u C Q = 2\pi\rho \frac{c}{2} u C[u\alpha + \dot{h} - (a-\frac{b}{2})\dot{\alpha}] \qquad \dot{\alpha} = 0.3\cos 15t$

$$= 2\pi(1.225)\tfrac{2}{2}(125)\, C[125(0.02\sin 15t) - (0.4 - \tfrac{2/2}{2})(0.3\cos 15t)]$$
$$= C[764.375\pi \sin 15t + 9.1725\pi\cos 15t]$$
$$= |C(k)|[764.375\pi\sin(15t + \text{Arg}\,C(k)) + 9.1725\pi\cos(15t + \text{Arg}\,C(k))]$$
$$= 1986.5\sin(15t - 11.9°) + 23.8\cos(15t - 11.9°)$$

(c) $\alpha(t) = 0.02\sin 150t, \quad \dot{\alpha}(t) = 3\cos 150t \qquad \omega = 150$ rad/s

$$k = \frac{150(2)}{2(125)} = 1.2$$
$$C(k) = 1 - \frac{0.165(1+91/2400 i)}{1+(91/2400)^2} - \frac{0.335(1+0.25i)}{1+0.25^2}$$
$$= 0.579942757 - 0.085070797 i$$

$|C(k)| = 0.526856252 = 0.527 \qquad \text{Arg}\, C(k) = -\tan^{-1}\frac{0.085}{0.5199} = -9.3°$

$$L_c = 2\pi(1.223)\tfrac{2}{2}(125)0.527[125(0.02\sin(150t - 9.3°) - (0.4 - \tfrac{2/2}{2})(3\cos(150t - 9.3°))]$$
$$= 1265.5\sin(150t - 9.3°) + 151.9\cos(150t - 9.3°)$$

The increased in frequency of oscillation shifts the phase of the lift lesser and less attenuates the lift magnitude. which can be seen from the lower magnitude of C(k) and lower phase of C(k)

4. (a) Assuming the solution in the form of $\begin{Bmatrix}\alpha\\ \theta\end{Bmatrix} = \begin{Bmatrix}\alpha\\ \theta\end{Bmatrix}_0 e^{\lambda t}$

$$\begin{vmatrix} 12\lambda^2 + 6V\lambda + (4\times10^5 - 9V^2) & 3V\lambda + 3V^2 \\ -3V^2 & \lambda^2 + V\lambda + V^2 \end{vmatrix} = 0$$

$a_{11} = 12$, $b_{11} = 6$, $c_{11} = -9$, $e_{11} = 4\times10^5 = xV^2$

$a_{12} = 0$, $b_{12} = 3$, $c_{12} = 3$

$a_{21} = 0$, $b_{21} = 0$, $c_{21} = -3$

$a_{22} = 1$, $b_{22} = 1$, $c_{22} = 1$, $e_{22} = 0 = \mu e_{11} \Rightarrow \mu = 0$

$b_4 = a_{11}a_{22} - a_{21}a_{12} = 12$

$b_3 = (a_{11}b_{22} + b_{11}a_{22} - a_{21}b_{12} - a_{12}b_{21})V = (12\cdot1 + 6\cdot1)V = 18V$

$b_2 = [(\mu a_{11} + a_{22})x + (a_{11}c_{22} + b_{11}b_{22} + c_{11}a_{22} - a_{21}c_{12} - b_{12}b_{21} - c_{21}a_{12})]V^2$

$= [x - (12+6-9)]V^2 = (x+9)V^2$

$b_1 = [(\mu b_{11} + b_{22})x + (b_{11}c_{22} + c_{11}b_{22} - b_{21}c_{12} - b_{12}c_{21})]V^3$

$= [x + (6-9+9)]V^3 = (x+6)V^3$

$b_0 = [\mu x^2 + (c_{22} + \mu c_{11})x + c_{11}c_{22} - c_{12}c_{21})]V^4$

$= [x - 9 + 9]V^4 = xV^4$

$b_4\lambda^4 + b_3\lambda^3 + b_2\lambda^2 + b_1\lambda + b_0 = 0$

$12\lambda^4 + 18V\lambda^3 + (x+9)V^2\lambda^2 + (x+6)V^3\lambda + xV^4 = 0$

$w_F^2 = \frac{b_1}{b_3} = \frac{(x+6)V^3}{18V} = \frac{(x+6)V^2}{18}$

$b_4w^4 - b_2w^2 + b_0 = 0$

$12\left[\frac{(x+6)V^2}{18}\right]^2 - [(x+9)V^2]\frac{(x+6)V^2}{18} + xV^4 = 0$

$\frac{1}{27}(x^2+12x+36) - \frac{1}{18}(x^2+15x+54) + x = 0$

$-\frac{1}{54}x^2 + \frac{11}{18}x - \frac{5}{3} = 0$

$x_1 = 30$, $x_2 = 3$

$xV^2 = 4\times10^5$

$x = x_1$, $V_{F1} = 115.47\ ms^{-1}$ $x = x_2$, $V_{F2} = 365.15\ ms^{-1}$

$w_{F1} = \sqrt{\frac{(30+6)(115.47)^2}{18}}$ $w_{F2} = \sqrt{\frac{(3+6)(365.15)^2}{18}}$

$= 163.30\ rads^{-1} = 25.99\ Hz$ $= 258.20\ rads^{-1} = 41.10\ Hz$

(b)(i) $U = \frac{1}{2}Kq_1^2 + \frac{1}{2}(2K)q_2^2$

Plate 1: $z = \frac{x}{L}q_1$ $\frac{dz}{dx} = \frac{q_1}{L}$

Plate 2: $z = (1-\frac{x}{L})q_1 + \frac{x}{L}q_2$ $\frac{dz}{dx} = \frac{q_2 - q_1}{L}$

Plate 3: $z = (1-\frac{x}{L})q_2$ $\frac{dz}{dx} = -\frac{q_2}{L}$

$T = \sum_{i=1}^{3}\frac{1}{2}\int_{x=0}^{L}\mu\left(\frac{\partial z_i}{\partial t}\right)^2 dx$

$= \frac{1}{2}\left[\int_{x=0}^{L}\mu(\frac{x}{L}\dot{q}_1)^2dx + \int_{x=0}^{L}\mu[(1-\frac{x}{L})\dot{q}_1 + \frac{x}{L}\dot{q}_2]^2dx + \int_{x=0}^{L}\mu[(1-\frac{x}{L})\dot{q}_2]^2dx\right]$

First term: $\frac{1}{2}\int_{x=0}^{L}\mu(\frac{x}{L}\dot{q}_1)^2dx = \frac{1}{2}\int_{x=0}^{L}\mu\frac{x^2}{L^2}\dot{q}_1^2dx$

$= \frac{\mu\dot{q}_1^2}{2L^2}\int_{x=0}^{L}x^2dx$

$= \frac{\mu\dot{q}_1^2}{2L^2}\left[\frac{x^3}{3}\right]_0^L = \frac{1}{6}\mu\dot{q}_1^2L$

Second term: $\frac{1}{2}\int_{x=0}^{L}\mu[(1-\frac{x}{L})\dot{q}_1 + \frac{x}{L}\dot{q}_2]^2dx = \frac{\mu}{2}\int_{x=0}^{L}[(1-\frac{x}{L})^2\dot{q}_1^2 + 2(1-\frac{x}{L})\frac{x}{L}\dot{q}_1\dot{q}_2 + \frac{x^2}{L^2}\dot{q}_2^2]dx$

$= \frac{\mu}{2}\int_{x=0}^{L}[(1-\frac{2x}{L}+\frac{x^2}{L^2})\dot{q}_1^2 + \frac{2x}{L}\dot{q}_1\dot{q}_2 - \frac{2x^2}{L^2}\dot{q}_1\dot{q}_2 + \frac{x^2}{L^2}\dot{q}_2^2]dx$

$= \frac{\mu}{2}\left[x\dot{q}_1^2 - \frac{x^2}{L}\dot{q}_1^2 + \frac{x^3}{3L^2}\dot{q}_1^2 + \frac{x^2}{L}\dot{q}_1\dot{q}_2 - \frac{2x^3}{3L^2}\dot{q}_1\dot{q}_2 + \frac{x^3}{3L^2}\dot{q}_2^2\right]_0^L$

$= \frac{\mu}{6}L\dot{q}_1^2 + \frac{\mu}{6}L\dot{q}_2^2 + \frac{\mu}{6}L\dot{q}_1\dot{q}_2$

Third term: $\frac{1}{2}\int_{x=0}^{L}\mu[(1-\frac{x}{L})\dot{q}_2]^2dx = \frac{\mu}{2}\int_{x=0}^{L}(1-\frac{x}{L})^2\dot{q}_2^2\,dx$

$= \frac{\mu}{2}\int_{x=0}^{L}(1-2\frac{x}{L}+\frac{x^2}{L^2})\dot{q}_2^2\,dx$

$= \frac{\mu}{2}[x\dot{q}_2^2 - \frac{x^2}{L}\dot{q}_2^2 + \frac{x^3}{3L^2}\dot{q}_2^2]_{x=0}^{L}$

$= \frac{\mu}{6}L\dot{q}_2^2$

$T = \frac{\mu L}{6}(2\dot{q}_1^2 + 2\dot{q}_2^2 + \dot{q}_1\dot{q}_2)$

$\delta W = \sum_{i=1}^{3}\int_{x=0}^{L}(-p)\,\delta z_i\,dx \quad \delta z_1 = \delta q_1\frac{x}{L} \quad \delta z_2 = \delta q_2(1-\frac{x}{L}) + \delta q_2\frac{x}{L} \quad \delta z_3 = \delta q_2(1-\frac{x}{L})$

$\delta W_1 = \int_{x=0}^{L}(-\frac{\rho V^2}{m}\frac{q_1}{L})\delta q_1\frac{x}{L}dx = -\frac{\rho V^2}{m}\frac{q_1}{L^2}\delta q_1[\frac{x^2}{2}]_0^L = -\frac{\rho V^2}{2m}q_1\delta q_1$

$\delta W_2 = \int_{x=0}^{L}(-\frac{\rho V^2}{2m}(\frac{q_2-q_1}{L}))[\delta q_2(1-\frac{x}{L}) + \delta q_2\frac{x}{L}]dx = -\frac{\rho V^2}{2m}(\frac{q_2-q_1}{L})[\delta q_1 x - \delta q_1\frac{x^2}{2L} + \delta q_2\frac{x^2}{2L}]_0^L$

$= -\frac{\rho V^2}{2m}(\frac{q_2-q_1}{L})(\delta q_1 L - \delta q_1\frac{L}{2} + \delta q_2\frac{L}{2})$

$= -\frac{\rho V^2}{4m}(q_2\delta q_1 + q_2\delta q_2 - q_1\delta q_1 - q_1\delta q_2)$

$= \frac{\rho V^2}{4m}(q_1\delta q_1 + q_1\delta q_2 - q_2\delta q_1 - q_2\delta q_2)$

$\delta W_3 = \int_{x=0}^{L}(\frac{\rho V^2}{m}\frac{q_2}{L})\delta q_2(1-\frac{x}{L})dx = \frac{\rho V^2}{m}\frac{q_2\delta q_2}{L}[x-\frac{x^2}{2L}]_0^L = \frac{\rho V^2}{m}\frac{q_2\delta q_2}{L}[L-\frac{L}{2}] = \frac{\rho V^2}{2m}q_2\delta q_2$

$\delta W = \sum_{i=1}^{3}\delta W_i = -\frac{\rho V^2}{2m}q_1\delta q_1 + \frac{\rho V^2}{4m}(q_1\delta q_1 + q_1\delta q_2 - q_2\delta q_1 - q_2\delta q_2) + \frac{\rho V^2}{2m}q_2\delta q_2$

$= -\frac{\rho V^2}{4m}q_1\delta q_1 + \frac{\rho V^2}{4m}q_1\delta q_2 - \frac{\rho V^2}{4m}q_2\delta q_1 + \frac{\rho V^2}{4m}q_2\delta q_2$

$= (-\frac{\rho V^2}{4m}(q_1+q_2))\delta q_1 + (\frac{\rho V^2}{4m}(q_1+q_2))\delta q_2$

$= Q_1\delta q_1 + Q_2\delta q_2$

$\frac{d}{dt}\left(\frac{\partial(T-U)}{\partial\dot{q}_1}\right) - \frac{\partial(T-U)}{\partial q_1} - Q_1 = 0$

$\frac{2}{3}\mu L\ddot{q}_1 + \frac{\mu L}{6}\ddot{q}_2 + kq_1 + \frac{\rho V^2}{4m}(q_1+q_2) = 0$

$\frac{d}{dt}\left(\frac{\partial(T-U)}{\partial\dot{q}_2}\right) - \frac{\partial(T-U)}{\partial q_2} - Q_2 = 0$

$\frac{2}{3}\mu L\ddot{q}_2 + \frac{\mu L}{6}\ddot{q}_1 + 2kq_2 - \frac{\rho V^2}{4m}(q_1+q_2) = 0$

$$\begin{bmatrix}\frac{2\mu L}{3} & \frac{\mu L}{6}\\ \frac{\mu L}{6} & \frac{2\mu L}{3}\end{bmatrix}\begin{Bmatrix}\ddot{q}_1\\ \ddot{q}_2\end{Bmatrix} + \begin{bmatrix}k+\frac{\rho V^2}{4m} & \frac{\rho V^2}{4m}\\ -\frac{\rho V^2}{4m} & 2k-\frac{\rho V^2}{4m}\end{bmatrix}\begin{Bmatrix}q_1\\ q_2\end{Bmatrix} = \begin{Bmatrix}0\\ 0\end{Bmatrix}$$

$$\mu L\begin{bmatrix}2/3 & 1/6\\ 1/6 & 2/3\end{bmatrix}\begin{Bmatrix}\ddot{q}_1\\ \ddot{q}_2\end{Bmatrix} + \left\{\frac{\rho V^2}{4m}\begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix} + \begin{bmatrix}k & 0\\ 0 & 2k\end{bmatrix}\right\}\begin{Bmatrix}q_1\\ q_2\end{Bmatrix} = \begin{Bmatrix}0\\ 0\end{Bmatrix}$$

Assuming $q_1 = \hat{q}_1 e^{j\omega t} \Rightarrow \ddot{q}_1 = -\omega^2 q_1$; $q_2 = \hat{q}_2 e^{j\omega t} \Rightarrow \ddot{q}_2 = -\omega^2 q_2$

$$\left[-\omega^2\mu L\begin{bmatrix}2/3 & 1/6\\ 1/6 & 2/3\end{bmatrix} + \begin{bmatrix}k & 0\\ 0 & 2k\end{bmatrix} + \frac{\rho V^2}{4m}\begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix}\right]\begin{Bmatrix}\hat{q}_1 e^{j\omega t}\\ \hat{q}_2 e^{j\omega t}\end{Bmatrix} = \begin{Bmatrix}0\\ 0\end{Bmatrix}$$

$$\begin{vmatrix}-\frac{2}{3}\omega^2\mu L + k + \frac{\rho V^2}{4m} & -\frac{1}{6}\omega^2\mu L + \frac{\rho V^2}{4m}\\ -\frac{1}{6}\omega^2\mu L - \frac{\rho V^2}{4m} & -\frac{2}{3}\omega^2\mu L + 2k - \frac{\rho V^2}{4m}\end{vmatrix} = 0$$

$(-\frac{2}{3}\omega^2\mu L + k + \frac{\rho V^2}{4m})(-\frac{2}{3}\omega^2\mu L + 2k - \frac{\rho V^2}{4m}) - (-\frac{1}{6}\omega^2\mu L - \frac{\rho V^2}{4m})(-\frac{1}{6}\omega^2\mu L + \frac{\rho V^2}{4m}) = 0$

$\frac{4}{9}\omega^4\mu^2L^2 - \frac{4}{3}k\omega^2\mu L + \frac{\rho V^2\omega^2\mu L}{6m} - \frac{2}{3}k\omega^2\mu L + 2k^2 - \frac{\rho kV^2}{4m} - \frac{\rho V^2\omega^2\mu L}{6} + \frac{\rho V^2 k}{2m} - \frac{\rho^2V^4}{16m^2} - (\frac{\omega^4\mu^2L^2}{36} - \frac{\rho^2V^4}{16m^2}) = 0$

$\frac{5}{12}\omega^4\mu^2L^2 - 2k\omega^2\mu L + \frac{\rho kV^2}{2m} + 2k^2 = 0$

$\frac{5}{12}\frac{\omega^4\mu^2L^2}{k^2} - 2\frac{\omega^2\mu L}{k} + \frac{\rho V^2}{2mk} + 2 = 0$

$\frac{5}{12}X^4 - 2X^2 + \lambda + 2 = 0$

$X^2 = \frac{2 \pm \sqrt{4 - 4(\frac{5}{12})(\lambda+2)}}{2(5/12)} = \frac{12}{5} \pm \sqrt{4 - \frac{5}{3}(\lambda+2)}$

Flutter occurs when $4 - \frac{5}{3}(\lambda+2) = 0$

$\lambda_F = 0.4$

$X_F = \sqrt{\frac{12}{5}}$

(ii) At natural frequency, $\lambda = 0$, $X^2 = \frac{12}{5} \pm \sqrt{\frac{2}{3}}$

$X_1 = 1.26$ $\quad X_2 = 1.79$

$X_1 = 1.26 = \sqrt{\frac{\mu L \omega^2}{K}}$ $\quad \lambda = 0 = \frac{\rho V^2}{2mE_1}$

$\omega^2 \mu L = 1.26^2 K$

$[-\frac{2}{3}(1.26^2 K) + K]\hat{q}_1 - \frac{1}{6}(1.26^2 K)\hat{q}_2 = 0$

$\left.\frac{\hat{q}_1}{\hat{q}_2}\right|_{x=x_1} = -0.22$

$X_2 = 1.79 = \sqrt{\frac{\mu L \omega^2}{K}}$

$\omega^2 \mu L = 1.79^2 K$

$[-\frac{2}{3}(1.79^2 K) + K]\hat{q}_1 - \frac{1}{6}(1.79^2) K \hat{q}_2 = 0$

$\left.\frac{\hat{q}_1}{\hat{q}_2}\right|_{x=x_2} = -2.13$

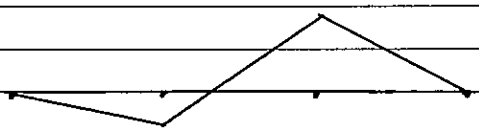
$X_f = \sqrt{\frac{12}{5}}$, $\omega^2 \mu L = \frac{12}{5} K$ $\quad \lambda = 0.4 = \frac{\rho V^2}{2mK}$

$\frac{\rho V^2}{4m} = 0.8K$

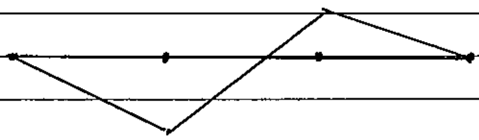
$[-\frac{2}{3}(\frac{12}{5}K) + K + 0.8K]\hat{q}_1 + (-\frac{1}{6}(\frac{12}{5}K) + 0.8K)\hat{q}_2 = 0$

$\left.\frac{\hat{q}_1}{\hat{q}_2}\right|_{x=x_f} = -0.5$

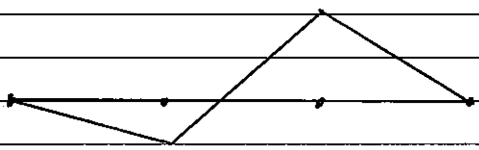
1st Mode



2nd Mode



Flutter mode



NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER 1 EXAMNINATION 2012-2013

AE4613 – AEROELASTICITY

November/December 2012 Time Allowed: 2 ½ hours

INSTRUCTIONS

1. This paper contains **FOUR** (4) questions and has **FOUR** (4) pages.
2. Answer **ALL** questions.
3. All questions carry equal marks.
4. This is an **Open-Book Examination**.

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1. A uniform string has a length $l=1m$ and a density $\rho=0.01kg/m$. It is fixed at both ends and it has a uniform initial tension $T=100N$.

   (a) State the governing equation of motion for string vibration and the assumptions made during the development of the equation.

   (5 marks)

   (b) In the case of free vibration, if the string is initially plucked as shown in Figure 1(a) with $h=1cm$ and is released at $t=0$, determine the subsequent vibration in terms of displacement $v(x,t)$.

   (10 marks)

   (c) In the case of forced vibration as shown in Figure 1(b), if a sinusoidal force $F_0=1N$ and $\Omega=10$ rad/s is applied to the initially stationary string in its equilibrium position, find the subsequent vibration of the string in terms of displacement.

   (10 marks)

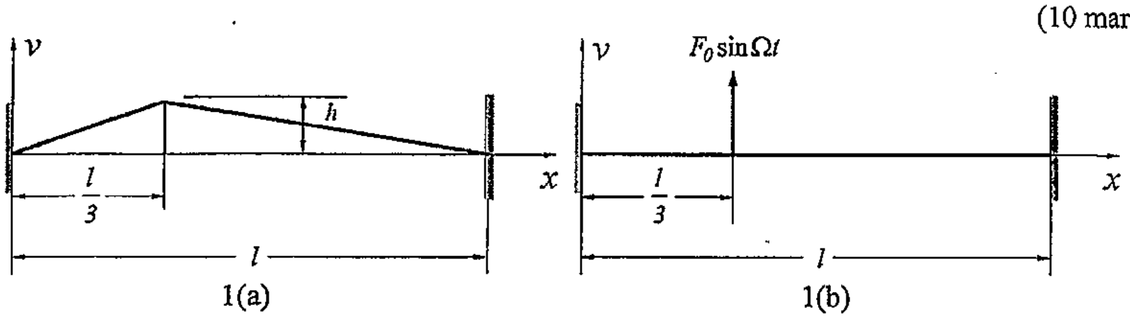


Figure 1 – A string with initial displacements (a) and under sinusoidal excitation (b)

2. A rigid model wing with aileron is elastically supported in a wind tunnel test, as shown in Figure 2. The torsional rigidity of the wing provides a torsional stiffness $k_T = 40,000 Nm/rad$. The wing is further constrained with springs of stiffness $2k$ at its leading edge and stiffness k at its trailing edge. Geometric parameters of the wing are: $S = 0.8m^2$, $c = 0.4m$, $e = 0.1m$, $a = 0.12m$, $d = 0.18m$. Aerodynamic coefficients are: $C_{L\alpha} = 6\ rad^{-1}$, $C_{M0} = 0.0\ rad^{-1}$, $C_{L\beta} = 1.4\ rad^{-1}$ and $C_{M\beta} = -0.4\ rad^{-1}$. Initial angle of attack α_r is assumed to be $\alpha_r = 3°$. Air density is assumed to be $\rho = 1.223\ kg/m^3$. At an aerodynamic pressure $q = 80,000 N/m^2$, aileron efficiency η is measured to be $\eta = 0.8$.

(a) Determine the value of spring constant k required.

(8 marks)

(b) Find the divergence speed U_D and reversal speed U_R of the system.

(7 marks)

(c) To further enhance the reversal speed by 5% by moving the pivot towards the trailing edge, find the new location x of the pivot.

(5 marks)

(d) For a dynamic pressure $q = 50,000 N/m^2$, compute the change in lift ΔL caused by the aeroelastic effect.

(5 marks)

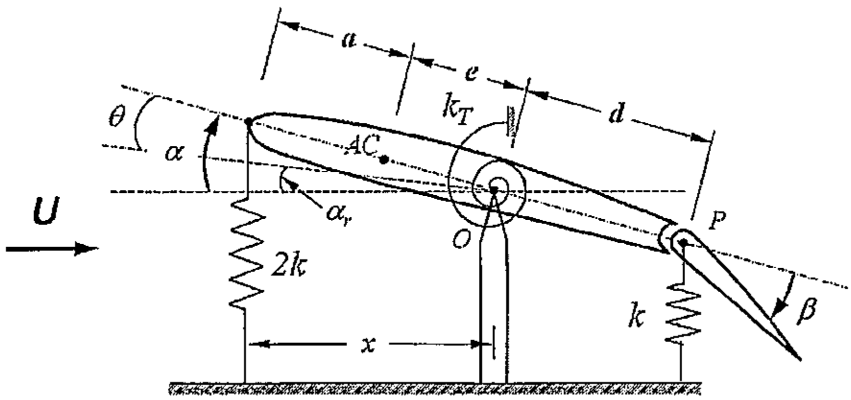


Figure 2 – An elastically restrained wing and its structural model

3. A small orbiter which can be modeled as an ideal flat wing, is suspended beneath its launch aircraft shown in Figure 3. The mass of the orbiter is $m = 100kg$. The mass moment of inertia of the orbiter with respect to its flexure centre A is $I_A = 3kg.m^2$. The distance between the mass centre O and the flexure centre A is $a = 0.1m$. The chord length is $c = 0.5m$. Other system parameters are: $k_1 = 25000N/m$, $k_2 = 10000N/m$ and $b = 0.12m$. Air density is assumed to be $\rho = 1.223kg/m^3$.

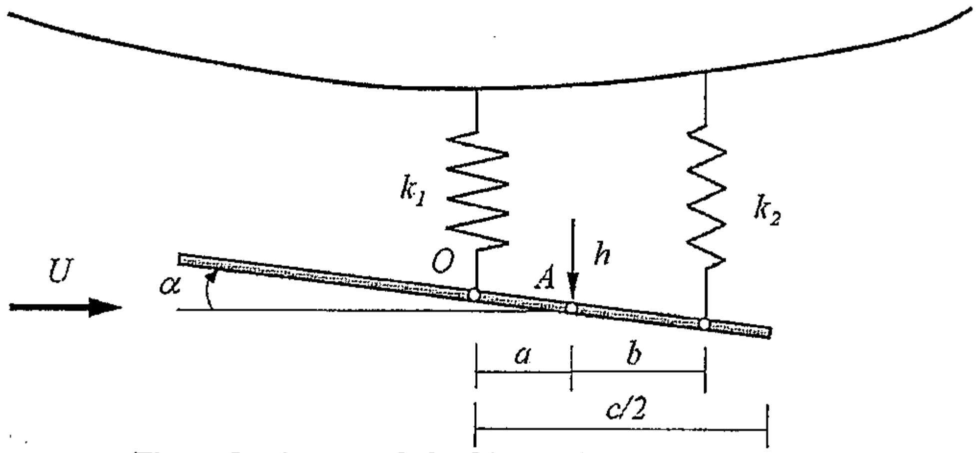


Figure 3 – A suspended orbiter and its flat wing model

(a) Derive the structural equations of motion (without the aeroelastic effect) in matrix form in terms of plunge degree of freedom h and pitch degree of freedom α.

(10 marks)

(b) For prescribed motions of the orbiter $h(t)=0$, $\alpha(t)=0.01\cos 80t$ and $U=100m/s$, find the magnitude and phase of the Theodorsen Function $C(k)$.

(8 marks)

(c) For the given motions of the orbiter described in (ii), find the complete lift and explain the physical meanings of the terms involved.

(7 marks)

4. A binary aeroelastic system (e.g. wing bending-torsion system) is modeled in terms of coordinates α and θ:

$$130\ddot{\theta}+6V\dot{\theta}+k\theta+3V^2\alpha=0$$

$$10\ddot{\alpha}-3V\dot{\theta}+V\dot{\alpha}+2k\alpha-3V^2\alpha=0$$

(a) Find the stiffness k that gives a critical flutter speed of V=250m/s.
(6 marks)

(b) Determine the corresponding flutter frequency (flutter speed of V=250 m/s)
(6 marks)

(c) Determine the divergence speeds and their corresponding frequencies.
(6 marks)

(d) If more than one value of stiffness k is estimated, there will be more than one flutter frequency corresponding to the flutter speed of V = 250 m/s. Identify which flutter frequency is more physically meaningful and explain why.
(7 marks)

End of Paper

1. (a) Governing equation of motion for string vibration : $\frac{\partial^2 v}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2 v}{\partial t^2}$

Assumptions: 1. Uniform string stretched between 2 fixed boundaries

2. Assume constant tension T in string

3. Small deformation in vertical displacement and slope

4. Tension initially set in string is so large that gravitational force can be neglected.

(b) For free vibration: $v(x,t) = \sum_{i=1}^{\infty} \sin\left(\frac{i\pi x}{\ell}\right)\left[E_i \sin\left(\sqrt{\frac{T}{\rho}}\frac{i\pi}{\ell}t\right) + F_i \cos\left(\sqrt{\frac{T}{\rho}}\frac{i\pi}{\ell}t\right)\right]$

where $E_i = \frac{2}{i\pi}\sqrt{\frac{\rho}{T}}\int_0^\ell g(x)\sin\left(\frac{i\pi x}{\ell}\right)dx \quad (i=1,2,\ldots) \qquad g(x) = \dot{v}(x,0)$

$F_i = \frac{2}{\ell}\int_0^\ell f(x)\sin\left(\frac{i\pi x}{\ell}\right)dx \quad (i=1,2,\ldots) \qquad f(x) = v(x,0)$

~~$0 \le x \le 1/3$~~ $x=0,\ f(0)=0$

$x = 1/3,\ f(1/3) = 0.01 \qquad f(x) = \frac{0.01x}{1/3} = 0.03x$

$1/3 < x < 1 \qquad \frac{x - 1/3}{1 - 1/3} = \frac{0.01 - f(x)}{0.01 - 0}$

$f(x) = -0.015(x - 1/3) + 0.01 = 0.015(1-x)$

$\therefore f(x) = \begin{cases} 0.03x & 0 \le x \le 1/3 \\ 0.015(1-x) & 1/3 < x \le 1.0 \end{cases}$

$g(x) = 0, \quad E_i = 0$

$F_i = \frac{2}{\ell}\int_0^\ell f(x)\sin\left(\frac{i\pi x}{\ell}\right)dx = 2\int_0^1 f(x)\sin(i\pi x)dx$

$= 2\int_0^{1/3} 0.03x\sin(i\pi x)dx + 2\int_{1/3}^1 0.015(1-x)\sin(i\pi x)dx$

$u = x,\ v' = \sin(i\pi x)$

$u' = 1,\ v = -\frac{\cos(i\pi x)}{i\pi}$

$F_i = 0.06\left[-\frac{x\cos(i\pi x)}{i\pi}\right]_0^{1/3} + 0.06\int_0^{1/3}\frac{\cos(i\pi x)}{i\pi}dx + 2\int_{1/3}^1 0.015\sin(i\pi x)dx - 2\int_{1/3}^1 0.015x\sin(i\pi x)dx$

$= 0.06\left[-\frac{\frac{1}{3}\cos(i\pi/3)}{i\pi} + 0\right] + \frac{0.06}{i\pi}\left[\frac{\sin(i\pi x)}{i\pi}\right]_0^{1/3} + 0.03\left[-\frac{\cos(i\pi x)}{i\pi}\right]_{1/3}^1 - 0.03\left[-\frac{x\cos(i\pi x)}{i\pi}\right]_{1/3}^1 + 0.03\int_{1/3}^1 -\frac{\cos(i\pi x)}{i\pi}dx$

$= -0.02\frac{\cos(i\pi/3)}{i\pi} + \frac{0.06}{i\pi}\left[\frac{\sin(i\pi/3)}{i\pi}\right] + 0.03\left[-\frac{(-1)^i}{i\pi} + \frac{\cos(i\pi/3)}{i\pi}\right] - 0.03\left[-\frac{(-1)^i}{i\pi} + \frac{1/3\cos(i\pi/3)}{i\pi}\right] - 0.03\left[\frac{\sin(i\pi x)}{(i\pi)^2}\right]_{1/3}^1$

$= -\frac{0.02}{i\pi}\cos\left(\frac{i\pi}{3}\right) + \frac{0.06}{(i\pi)^2}\sin\left(\frac{i\pi}{3}\right) - \frac{0.03}{i\pi}(-1)^i + \frac{0.03}{i\pi}\cos\left(\frac{i\pi}{3}\right) + \frac{0.03}{i\pi}(-1)^i - \frac{0.01}{i\pi}\cos\left(\frac{i\pi}{3}\right) + \frac{0.03}{(i\pi)^2}\sin\left(\frac{i\pi}{3}\right)$

$= \frac{0.09}{(i\pi)^2}\sin\left(\frac{i\pi}{3}\right)$

$\phi_i = \sin\left(\frac{i\pi x}{\ell}\right) = \sin(i\pi x)$

$w_i = \sqrt{\frac{T}{\rho}}\frac{i\pi}{\ell} = \sqrt{\frac{100}{0.01}}\frac{i\pi}{1} = 100i\pi$

$\therefore v(x,t) = \sum_{i=1}^{\infty}\sin(i\pi x)\left[\frac{0.09}{(i\pi)^2}\sin\left(\frac{i\pi}{3}\right)\cos(100i\pi t)\right] = \frac{0.09}{\pi^2}\sum_{i=1}^{\infty}\frac{1}{i^2}\sin\left(\frac{i\pi}{3}\right)\sin(i\pi x)\cos(100i\pi t)$ *

(c) $\theta_i(t) = \int_0^\ell F(x,t)\phi_i(x)dx = \int_0^\ell F_0\,\delta(x - x_1)\sin\left(\frac{i\pi x}{\ell}\right)dx$

$= \int_0^1 \delta(x - 1/3)\sin(i\pi x)dx = \sin\left(\frac{i\pi}{3}\right)$

$\frac{\rho\ell}{2}\ddot{\xi}_i(t) + \frac{\rho\ell}{2}w_i^2\xi_i(t) = \theta_i$

$\ddot{\xi}_i(t) + w_i^2\xi_i(t) = \frac{2}{\rho\ell}\theta_i = \frac{2}{0.01\cdot 1}\sin\left(\frac{i\pi}{3}\right) \qquad w_i = \frac{i\pi}{\ell}\sqrt{\frac{T}{\rho}} = 100i\pi$

$\ddot{\xi}_i(t) + (100i\pi)^2\xi_i(t) = 200\sin\left(\frac{i\pi}{3}\right)$

Homogeneous solution : $\xi_h(t) = A_i\sin(100i\pi)t + B_i\cos(100i\pi)t$

Particular solution : $\xi_p(t) = c,\ \ddot{\xi}_p(t) = 0$

$(100i\pi)^2 c = 200\sin\left(\frac{i\pi}{3}\right)$

$c = \frac{0.02}{(i\pi)^2}\sin\left(\frac{i\pi}{3}\right)$

$\therefore \xi_i(t) = A_i\sin(100i\pi)t + B_i\cos(100i\pi)t + \frac{0.02}{(i\pi)^2}\sin\left(\frac{i\pi}{3}\right)$

$v(x,t) = \sum_{i=1}^{\infty}\phi_i(x)\xi_i(t)$

$= \sum_{i=1}^{\infty}\sin(i\pi x)\left[A_i\sin(100i\pi)t + B_i\cos(100i\pi)t + \frac{0.02}{(i\pi)^2}\sin\left(\frac{i\pi}{3}\right)\right]$

$\dot{v}(x,t) = \sum_{i=1}^{\infty}\phi_i(x)\dot{\xi}_i(t) = \sum_{i=1}^{\infty}\sin(i\pi x)\left[A_i(100i\pi)\cos(100i\pi)t - B_i(100i\pi)\sin(100i\pi)t\right]$

$\dot{v}(x,0) = 0 \Rightarrow \sum_{i=1}^{\infty}\sin(i\pi x)A_i(100i\pi) = 0 \Rightarrow A_i = 0$

$$v(x,0)=0 \Rightarrow \sum_{i=1}^{\infty} \sin(i\pi x)\left[B_i + \frac{0.02}{(i\pi)^2}\sin\left(\frac{i\pi}{3}\right)\right]=0$$

$$B_i + \frac{0.02}{(i\pi)^2}\sin\left(\frac{i\pi}{3}\right)=0$$

$$B_i = -\frac{0.02}{(i\pi)^2}\sin\left(\frac{i\pi}{3}\right)$$

$$\therefore v(x,t) = \sum_{i=1}^{\infty} \sin(i\pi x)\left[-\frac{0.02}{(i\pi)^2}\sin\left(\frac{i\pi}{3}\right)\cos(100i\pi)t + \frac{0.02}{(i\pi)^2}\sin\left(\frac{i\pi}{3}\right)\right]$$

$$= \frac{0.02}{\pi^2}\sum_{i=1}^{\infty}\frac{\sin(i\pi x)}{i^2}\sin\left(\frac{i\pi}{3}\right)(1-\cos(100i\pi)t) \ \divideontimes$$

2.(a) $\eta = \dfrac{1-q/q_R}{1-q/q_D}$

$$0.8 = \frac{1-80000/q_R}{1-80000/q_D}$$

Moment equilibrium about Elastic Axis:

$$M_{ac} + Le = k_T\theta + \theta x(2k)x + \theta d(k)(d)$$

$$= (k_T + 2kx^2 + kd^2)\theta$$

Equivalent torsional stiffness of the system, $\bar{k} = k_T + 2kx^2 + kd^2$

$$= 40000 + 0.1292k$$

$$q_R = -\frac{\bar{k}C_{L\beta}}{C_{L\alpha}ScC_{m\beta}} = -\frac{(40000+0.1292k)(1.4)}{6(0.8)(0.4)(-0.4)} = \frac{218750}{3} + \frac{2261}{9600}k$$

$$q_D = \frac{\bar{k}}{SC_{L\alpha}e} = \frac{(40000+0.1292k)}{0.8(6)(0.1)} = \frac{250000}{3} + \frac{323}{1200}k$$

$$\eta = 0.8 = \frac{1-\dfrac{80000}{218750/3+2261/9600k}}{1-\dfrac{80000}{250000/3+323/1200k}}$$

$$\frac{80000\left(\frac{250000}{3}+\frac{323}{1200}k\right)-64000\left(\frac{218750}{3}+\frac{2261}{9600}k\right)}{\left(\frac{218750}{3}+\frac{2261}{9600}k\right)\left(\frac{250000}{3}+\frac{323}{1200}k\right)}=0.2$$

$$\frac{2000M + 6460k}{6076388889 + \frac{1413125}{36}k + 0.063394357k^2} = 0.2$$

$$0.012678871k^2 + \frac{50065}{36} - 7847222222.2 = 0$$

$$k = 199{,}911.5468 \approx 200k \text{ Nm/rad} \quad ; \quad k = -310k \text{ (rejected)}$$

(b) $\bar{k} = 65828.57185$ Nm/rad

$$q_D = \frac{\bar{k}}{SC_{L\alpha}e} = \frac{65828.6}{0.8(6)(0.1)} = 137142.858 \text{ N/m}^2$$

$$U_D = \sqrt{\frac{2q_D}{\rho}} = 473.6 \text{ ms}^{-1} \ \divideontimes$$

$$q_R = -\frac{\bar{k}C_{L\beta}}{C_{L\alpha}ScC_{m\beta}} = -\frac{65828.6(1.4)}{6(0.8)(0.4)(-0.4)} = 120000 \text{ N/m}^2$$

$$U_R = \sqrt{\frac{2q_R}{\rho}} = 443.0 \text{ ms}^{-1} \ \divideontimes$$

(c) $e' = x - a = x - 0.12$

$$U_{R,new} = 1.05U_R = 465.1 \text{ ms}^{-1}$$

$$q_R = \frac{1}{2}\rho U_{R,new}^2 = 132300 \text{ N/m}^2$$

$$\bar{k} = k_T + 2kx^2 + kd^2 = 40000 + 2(200k)x^2 + 200k(c-x)^2 \qquad c=0.4$$

$$= 40000 + 400kx^2 + 32k - 160kx + 200kx^2$$

$$-\frac{(72000-160kx+600kx^2)C_{L\beta}}{C_{L\alpha}ScC_{m\beta}} = 132300$$

$$-\frac{(72000-160kx+600kx^2)(1.4)}{6(0.8)(0.4)(-0.4)} = 132300$$

$$600kx^2 - 160kx - 576 = 0$$

$$x = 0.27\text{m} \quad , \quad x = -3.55\times10^{-3}\text{m (rejected)}$$

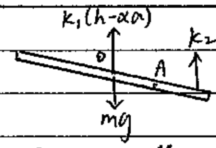
(d) $\Delta L = C_{L\beta}\left(1-\dfrac{q}{q_R}\right)$

$$= 1.4\left(1-\frac{50000}{120000}\right)$$

$$= 0.82$$

$$= 82\%$$

3. (a)



$\Sigma F = ma_A$

$mg - k_1(h-\alpha a) - k_2(h+\alpha b) = m\ddot{h}$

$m\ddot{h} + (k_1+k_2)h - (k_1 a + k_2 b)\alpha - mg = 0$

$\Sigma M_A = I_A \ddot{\alpha}$

$k_1(h-\alpha a)a - k_2(h+\alpha b)b - mga = I_A\ddot{\alpha}$

$I_A\ddot{\alpha} + (k_1a^2 + k_2b^2)\alpha - (k_1a+k_2b)h + mga = 0$

$$\begin{bmatrix} m & 0 \\ 0 & I_A \end{bmatrix}\begin{Bmatrix}\ddot{h}\\ \ddot{\alpha}\end{Bmatrix} + \begin{bmatrix} k_1+k_2 & -k_1a-k_2b \\ -k_1a-k_2b & k_1a^2+k_2b^2\end{bmatrix}\begin{Bmatrix} h \\ \alpha\end{Bmatrix} = \begin{bmatrix} mg \\ -mga\end{bmatrix}$$

Substituting known parameters, $\begin{bmatrix}100 & 0\\ 0 & 3\end{bmatrix}\begin{Bmatrix}\ddot{h}\\ \ddot{\alpha}\end{Bmatrix} + \begin{bmatrix}35000 & -3700\\ -3700 & 394\end{bmatrix}\begin{Bmatrix}h\\ \alpha\end{Bmatrix} = \begin{bmatrix}981\\ -98.1\end{bmatrix}$

(b) $\omega = 80\,rad/s$, $k = \frac{\omega b}{u} = \frac{\omega c}{2u} = \frac{80\times 0.5}{2\times 100} = 0.2$

$C(k) = 1 - \frac{0.165}{1-(0.0455/k)i} - \frac{0.335}{1-(0.30/k)i}$

$= 1 - \frac{0.165}{1-0.2275i} - \frac{0.335}{1-1.5i}$

$= 1 - \frac{0.165(1+0.2275i)}{1^2+0.2275^2} - \frac{0.335(1+1.5i)}{1^2+1.5^2}$

$= 0.74 - 0.19i$

$|C(k)| = 0.764$ $\quad \arg C(k) = -\tan^{-1}\frac{0.19}{0.74} = -14.4^\circ$

(c) $L_{nc} = \rho\pi b^2[U\dot{\alpha} - a\ddot{\alpha} + \ddot{h}]$ $\quad \dot{\alpha} = -0.8\sin 80t$ $\quad \ddot{\alpha} = -0.64\cos 80t$

$= 1.223\pi\left(\frac{0.5}{2}\right)^2[100(-0.8\sin 80t) - 0.1(-0.64\cos 80t)]$

$= (4.892\times 10^{-3})\pi\cos 80t - 6.115\pi\sin 80t \quad N/m$

$L_c = 2\pi\rho b U C Q$

$= 2\pi\rho b U C[U\alpha + \dot{h} - (a - \frac{b}{2})\dot{\alpha}]$

$= 2\pi(1.223)(\frac{0.5}{2})(100)C(k)[100(0.01\cos 80t) - (0.1 - \frac{0.5}{4})(-0.8\sin 80t)]$

$= 61.15\pi C(k)[\cos 80t - 0.02\sin 80t]$

$= 61.15\pi|C(k)|[\cos(80t + \mathrm{Arg}\,C(k)) - 0.02\sin(80t + \mathrm{Arg}\,C(k))]$

$= 46.7186\pi[\cos(80t - 14.4^\circ) - 0.02\sin(80t - 14.4^\circ)]$

$L = L_{nc} + L_c = 15369\cos 80t - 19.2\sin 80t + 146.8\cos(80t - 14.4^\circ) - 29\sin(80t - 14.4^\circ)$

L_{nc} is generated due to the acceleration $\ddot{h}$ of the plunge degree of freedom, the angular acceleration $\ddot{\alpha}$ of the pitch degree of freedom and the coriolis acceleration $h\dot{\alpha}$. It is an inertial force which leads to an additional added mass to the system. L_c is generated due to circulation around airfoil, which is caused by wake vortex due to pitching and heaving.

4. (a) $\begin{bmatrix}130 & 0\\ 0 & 10\end{bmatrix}\begin{Bmatrix}\ddot{\theta}\\ \ddot{\alpha}\end{Bmatrix} + V\begin{bmatrix}6 & 0\\ -3 & 1\end{bmatrix}\begin{Bmatrix}\dot{\theta}\\ \dot{\alpha}\end{Bmatrix} + \left\{V^2\begin{pmatrix}0 & 3\\ 0 & -3\end{pmatrix} + \begin{pmatrix}k & 0\\ 0 & 2k\end{pmatrix}\right\}\begin{Bmatrix}\theta\\ \alpha\end{Bmatrix} = \begin{Bmatrix}0\\ 0\end{Bmatrix}$

$a_{11} = 130$ $\quad a_{12} = 0$ $\quad a_{21} = 0$ $\quad a_{22} = 10$

$b_{11} = 6$ $\quad b_{12} = 0$ $\quad b_{21} = -3$ $\quad b_{22} = 1$

$c_{11} = 0$ $\quad c_{12} = 3$ $\quad c_{21} = 0$ $\quad c_{22} = -3$

$e_{11} = k$ $\quad e_{22} = 2k$ $\quad e_{11} = xV^2 = k$ $\quad e_{22} = \mu e_{11} = \mu x V^2 = 2k \Rightarrow \mu = 2$

$b_4 = (a_{11}a_{22} - a_{21}a_{12}) = 1300$

$b_3 = (a_{11}b_{22} + b_{11}a_{22} - a_{21}b_{12} - a_{12}b_{21})V = 190V$

$b_2 = [(\mu a_{11} + a_{22})x + (a_{11}c_{22} + b_{11}b_{22} + c_{11}a_{22} - a_{21}c_{12} - b_{12}b_{21} - c_{21}a_{12})]V^2 = (270x - 384)V^2$

$b_1 = [(\mu b_{11} + b_{22})x + (b_{11}c_{22} + c_{11}b_{22} - b_{21}c_{12} - b_{12}c_{21})]V^3 = (13x - 9)V^3$

$b_0 = [\mu x^2 + (C_{22} + \mu C_{11})x + C_{11}C_{22} - C_{12}C_{21}]V^4 = (2x^2 - 3x)V^4$

$b_4\lambda^4 + b_3\lambda^3 + b_2\lambda^2 + b_1\lambda + b_0 = 0$

$1300\lambda^4 + 190V\lambda^3 + (270x - 384)V^2\lambda^2 + (13x - 9)V^3\lambda + (2x^2 - 3x)V^4 = 0$

$\omega_F^2 = \frac{b_1}{b_3} = \frac{(13x-9)V^2}{190}$

$b_4\omega^4 - b_2\omega^2 + b_0 = 0$

$1300\omega^4 - (270x - 384)V^2\omega^2 + (2x^2 - 3x)V^4 = 0.$

$1300\left[\frac{(13x-9)V^2}{190}\right]^2 - (270x - 384)V^2\left(\frac{(13x-9)V^2}{190}\right) + (2x^2 - 3x)V^4 = 0$

$-\frac{3750}{361}x^2 + \frac{49884}{1805}x - 15.27257618 = 0$

$x_1 = 0.783158235 \qquad x_2 = 1.877321765$

$xV^2 = k$

$x = x_1 \Rightarrow k = 48.9 \text{ kNm}^{-1} \qquad x = x_2 \Rightarrow k = 117.3 \text{ kNm}^{-1}$

(b) $\omega_F^2 = \frac{(13x-9)V^2}{190}$

$x = x_1 \Rightarrow \omega_F = 19.7 \text{ rads}^{-1} = 3.14 \text{ Hz}$

$x = x_2 \Rightarrow \omega_F = 71.2 \text{ rads}^{-1} = 11.33 \text{ Hz}$

(c) $\left\{V^2\begin{pmatrix}0 & 3\\ 0 & -3\end{pmatrix} + \begin{pmatrix}k & 0\\ 0 & 2k\end{pmatrix}\right\}\begin{Bmatrix}\theta\\ \alpha\end{Bmatrix} = \begin{Bmatrix}0\\ 0\end{Bmatrix}$

$\begin{vmatrix}k & 3V^2\\ 0 & -3V^2 + 2k\end{vmatrix} = 0$

$k(-3V^2 + 2k) = 0$

$V = \sqrt{\frac{2k}{3}}$

$k = 48.9k, \quad V = 180.6 \text{ ms}^{-1}$

$k = 117.3k, \quad V = 279.6 \text{ ms}^{-1}$

Since divergence is a static phenomenon, there is no divergence frequencies.

(d). The lower frequency which is 3.14 Hz is more physically meaningful. This is because the system will increase its frequency from zero. Once it hits the lower frequency, the structure will undergo flutter and hence structural failure. Thus it would be unable to further continue functioning to a higher frequency after that.

NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER 2 EXAMINATION 2013-2014

AE4613/MA4704 – AEROELASTICITY

April/May 2014 Time Allowed: 2 ½ hours

INSTRUCTIONS

1. This paper contains **FOUR (4)** questions and has **FOUR (4)** pages.
2. Answer **ALL** questions.
3. All questions carry equal marks.
4. This is an **OPEN-BOOK Examination.**

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1. A torsionally elastic ($GI_P = 80\ Nm^2$) wind tunnel model of a uniform wing whose ends are pre-twisted elastically and then fastened rigidly to wind tunnel walls to effect such boundary conditions as $\theta(0) = 0$ at $y = 0$ and $\theta(l) = -3^\circ$ at $y = l$, is shown in Figure 1. The model has symmetric airfoil, a span of $1m$, and a chord of $0.16m$. The section lift curve slope is 5 per rad. The aerodynamic centre is located at the quarter chord and both the mass centre and the elastic axis are at the mid-chord. The rigid initial angle of attack $a_r = 3^\circ$.

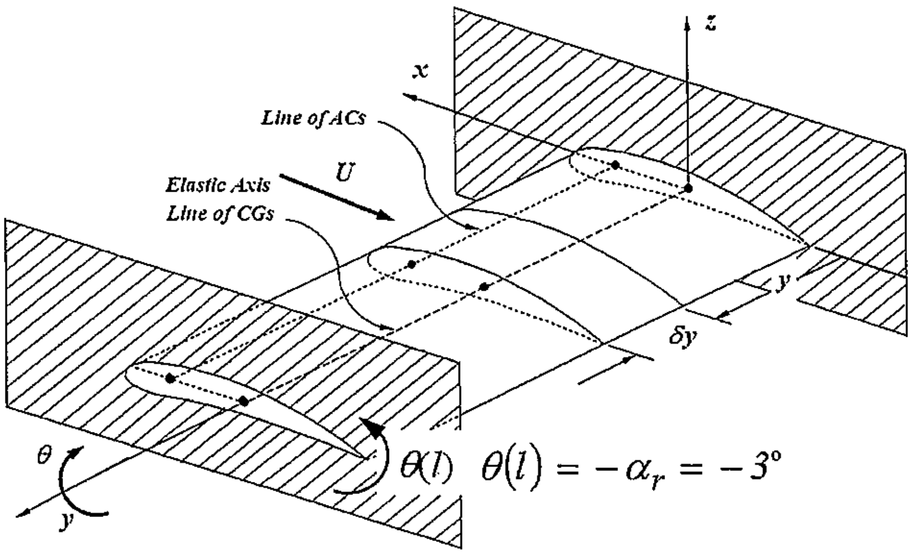

Figure 1

Note: Question 1 continues on page 2.

(a) Derive the governing equation for the torsional displacement $\theta(y)$.

(5 marks)

(b) Solve the equation to obtain the required $\theta(y)$ by imposing the given boundary conditions.

(9 marks)

(c) Derive the lift distribution $L'(y)$ and its value when $q = 5000\text{N/m}^2$, $y = 0.5\text{m}$.

(5 marks)

(d) Determine the divergence speed of the wing.

(6 marks)

2. A uniform string has a length $l = 1m$, a density $\rho = 0.02kg/m$. The string is fixed at both ends with initial tension set at $T = 200N$. Two lumped masses of 4 gram each are added to the string at the locations as shown in Figure 2. The effect of a lumped mass m can be considered as an external inertial force applied to the uniform string $F = -m\ddot{v}$ where v is the displacement at the location where the said mass is attached.

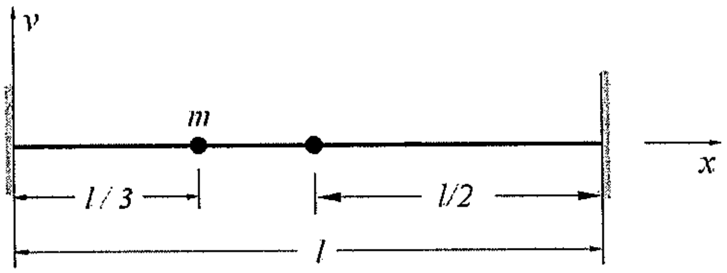

Figure 2

(a) Write the governing equation of motion and determine the natural frequencies and mode shapes of the uniform string.

(6 marks)

(b) Using the lowest 2 modes of the uniform string, derive the reduced 2 equations of motion of the modified string with 2 lumped masses and establish the 2×2 mass and stiffness matrices [M] and [K].

(12 marks)

(c) Find the two natural frequencies of the modified string and comment on their values when compared with those of the uniform string.

(7 marks)

3. An aircraft wing has a torsional stiffness $k_T = 2500$ Nm and a bending stiffness $k = 5$ MN/m as shown in Figure 3. The wing is assumed to have a symmetric airfoil ($C_{Mac} = 0$) with a span of 1m, a chord of 0.16 m and lift slope of $C_{L\alpha} = 6$ per rad. The elastic axis A is located at mid-chord and the aerodynamic centre ac is at quarter chord. Air density is assumed to be $\rho = 1.223\text{kg/m}^3$. Wing torsional stiffness can be further modified through a linear spring $\bar{k}$ attached at its trailing edge. The torsional spring k_T and the linear springs k and $\bar{k}$ are assumed to be undeformed when $h = 0$ and $\alpha = 0$.

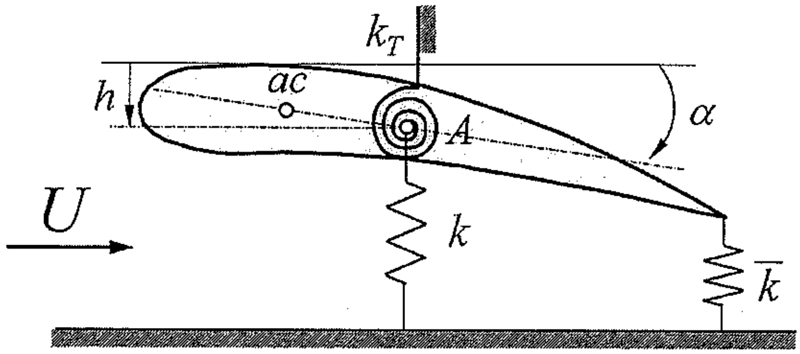

Figure 3

(a) Assume $\bar{k} = 0\ N/m$, in the case of static displacements under steady flow, derive the equations of motion in terms of h and α.

(4 marks)

(b) Derive the torsional divergence speed and show that it is independent of the bending stiffness k of the wing.

(5 marks)

(c) Assume $\bar{k} = 100{,}000 N/m$, in the case of static displacements under steady flow, derive the equations of motion in terms of h and α.

(8 marks)

(d) Compute the new divergence speed.

(8 marks)

Hint: Divergence speed can be obtained by setting the determinant of the coefficient matrix of the equations you establish in terms of h and α to zero.

If $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{Bmatrix} h \\ \alpha \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$, then, $\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{21}a_{12} = 0.$

4. A binary aeroelastic system (e.g. wing bending-torsion system) is modeled in terms of coordinates γ and θ:

$$\begin{bmatrix} 160 & 0 \\ 0 & 20 \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{\gamma} \end{bmatrix} + \begin{bmatrix} 5V & 0 \\ -2V & V \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \dot{\gamma} \end{bmatrix} + \begin{bmatrix} k & 2V^2 \\ 0 & 4k-2V^2 \end{bmatrix} \begin{bmatrix} \theta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

If $k = 2.5 \times 10^4$ N/m, then

(a) Determine the critical flutter speeds.

(9 marks)

(b) Determine the corresponding flutter frequencies

(9 marks)

(c) If there are multiple flutter frequencies, discuss which flutter frequency is more physically meaningful and explain why.

(7 marks)

END OF PAPER

1. (a) $\Delta L = qc\delta y C_{l\alpha}(\alpha_r - \bar{\theta})$, $\Delta M_{ac} = qc^2 \delta y C_{mac} = 0$ $\theta = -\bar{\theta}$

$\Sigma M_{fc} = 0$

$\Delta M_{ac} - T - \frac{dT}{dy}\delta y + \Delta L e + T - Npg\delta y d = 0$

$qc^2\delta y C_{mac} - \frac{dT}{dy}\delta y + qc\delta y C_{l\alpha}(\alpha_r - \bar{\theta})e - Npg\delta y d = 0$

$qc^2 C_{mac} - \frac{dT}{dy} + qcC_{l\alpha}(\alpha_r - \bar{\theta})e - Npgd = 0$

$T = GI_p \frac{d\theta}{dy}$: $qc^2 C_{mac} + GI_p \frac{d^2\theta}{dy^2} + qcC_{l\alpha}(\alpha_r + \theta)e - Npgd = 0$

$\frac{d^2\theta}{dy^2} + \frac{qecC_{l\alpha}}{GI_p}\theta = -\frac{1}{GI_p}(qc^2 C_{mac} + qecC_{l\alpha}\alpha_r - Npgd)$

C_{mac} & $d = 0$, $\frac{d^2\theta}{dy^2} + \frac{qecC_{l\alpha}}{GI_p}\theta = -\frac{1}{GI_p}(qecC_{l\alpha}\alpha_r)$

(b) Homogeneous solution : $\theta(y) = A\cos(\lambda y) + B\sin(\lambda y)$ $\lambda = \sqrt{\frac{qecC_{l\alpha}}{GI_p}}$

Particular solution : let $\theta_p(y) = C$ $\ddot{\theta}_p(y) = 0$

$\frac{qecC_{l\alpha}}{GI_p} C = -\frac{qecC_{l\alpha}\alpha_r}{GI_p}$

$C = -\alpha_r$

$\therefore \theta(y) = A\cos(\lambda y) + B\sin(\lambda y) - \alpha_r$

The boundary conditions are : $\theta = 0$ when $y = 0$; $\theta = -3°$ when $y = l$

$\theta = 0$ when $y = 0$: $A - \alpha_r = 0 \Rightarrow A = \alpha_r$

$\theta = -3°$ when $y = l$: $A\cos(\lambda l) + B\sin(\lambda l) - \alpha_r = \frac{-3\times\pi}{180}$

$\alpha_r\cos(\lambda l) + B\sin(\lambda l) - \alpha_r = -\frac{\pi}{60}$

$B = \frac{\alpha_r - \alpha_r\cos(\lambda l) - \pi/60}{\sin(\lambda l)} = \alpha_r\tan\left(\frac{\lambda l}{2}\right) - \frac{\pi}{60\sin(\lambda l)}$

$\therefore \theta(y) = \alpha_r\cos(\lambda y) + \left[\alpha_r\tan\left(\frac{\lambda l}{2}\right) - \frac{\pi}{60\sin(\lambda l)}\right]\sin(\lambda y) - \alpha_r$

(c) $L' = qcC_{l\alpha}(\alpha_r + \theta)$

$= qcC_{l\alpha}\left(\alpha_r + \alpha_r\cos(\lambda y) + \left[\alpha_r\tan\left(\frac{\lambda l}{2}\right) - \frac{\pi}{60\sin(\lambda l)}\right]\sin(\lambda y) - \alpha_r\right)$

when $q = 5000\ N/m^2$ & $y = 0.5m$, $\lambda = \sqrt{\frac{5000(0.16/4)(0.16)(5)}{80}} = \sqrt{2} = 1.414$

$L' = (5000)(0.16)(5)\left[\frac{3\pi}{180}\cos[\sqrt{2}\cdot 0.5] + \left[\frac{3\pi}{180}\tan\left(\frac{\sqrt{2}\cdot 1}{2}\right) - \frac{\pi}{60\sin(\sqrt{2}\cdot 1)}\right]\sin(\sqrt{2}\cdot 0.5)\right]$

$= 82.0\ N/m$

(d) Divergence occurs when $\frac{\lambda l}{2} = \frac{\pi}{2} \Rightarrow \lambda l = \pi$

$\sqrt{\frac{qecC_{l\alpha}}{GI_p}}\, l = \pi$

$q_D = \frac{\pi^2}{l^2}\frac{GI_p}{ecC_{l\alpha}} = \frac{\pi^2}{1^2}\frac{80}{\frac{0.16}{4}(0.16)(5)} = 24674.011\ N/m^2$

Divergence speed, $u_D = \sqrt{\frac{2q_D}{\rho}} = \sqrt{\frac{2(24674.011)}{1.223}} = 200.87\ m/s$

2. Out of syllabus.

3. (a) $L = qSC_L = qSC_{l\alpha}\alpha$ $M_{ac} = 0$

$\Sigma F = 0$, $-L + kh = 0$

$qSC_{l\alpha}\alpha + kh = 0$

$\frac{1}{2}\rho u^2 bcC_{l\alpha}\alpha + kh = 0$

$\Sigma M = 0$, $L\cos\alpha e - k\alpha = 0$

$qSC_{l\alpha}\alpha\cos\alpha e - k\alpha = 0$

$\alpha \approx 0$, $\cos\alpha = 1$ $\therefore qSC_{l\alpha}\alpha e - k_T\alpha = 0$

$\frac{1}{2}\rho u^2 bcC_{l\alpha}\alpha e - k_T\alpha = 0$

Equation of motion : $\begin{bmatrix} k & \frac{1}{2}\rho u^2 bcC_{l\alpha}\alpha + kh \\ 0 & \frac{1}{2}\rho u^2 bcC_{l\alpha}e - k_T \end{bmatrix}\begin{Bmatrix} h \\ \alpha \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$

(b) Divergence occurs at the non-trivial solution defined by

$$\begin{vmatrix} k & \frac{1}{2}\rho u^2 bcC_{L\alpha}\alpha + kh \\ 0 & \frac{1}{2}\rho u^2 bcC_{L\alpha}e - k_T \end{vmatrix} = 0$$

$$k(\tfrac{1}{2}\rho u^2 bcC_{L\alpha}e - k_T) = 0$$

Since $k \neq 0$, $\frac{1}{2}\rho u^2 bcC_{L\alpha}e - k_T = 0$

$$u_D = \sqrt{\frac{2k_T}{\rho bcC_{L\alpha}e}}$$, it is independent of k

$$= \sqrt{\frac{2(2500)}{(1.223)(1)(0.16)(6)(\frac{0.16}{4})}} = 326.3\ ms^{-1}$$

(c) $\Sigma F = 0$, $-L + kh + \bar{k}(h + \frac{c\alpha}{2}) = 0$

$$qSC_{L\alpha}\alpha + (k + \bar{k})h + \frac{\bar{k}}{2}c\alpha = 0$$

$$(k + \bar{k})h + (qSC_{L\alpha} + \frac{\bar{k}}{2}c)\alpha = 0$$

$\Sigma M = 0$ $\quad qSC_{L\alpha}\alpha e - k_T\alpha - \bar{k}(\frac{c}{2})^2\alpha = 0$

$$(qSC_{L\alpha}e - k_T - \bar{k}\frac{c^2}{4})\alpha = 0$$

$$\begin{bmatrix} k+\bar{k} & \frac{1}{2}\rho u^2 bcC_{L\alpha} + \frac{\bar{k}}{2}c \\ 0 & \frac{1}{2}\rho u^2 bcC_{L\alpha}e - k_T - \bar{k}\frac{c^2}{4} \end{bmatrix} \begin{Bmatrix} h \\ \alpha \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

(d) $(k+\bar{k})(\frac{1}{2}\rho u^2 bcC_{L\alpha}e - k_T - \bar{k}\frac{c^2}{4}) = 0$

$$\tfrac{1}{2}(1.223)u_D^2(1)(0.16)(6)(\tfrac{0.16}{4}) - 2500 - 100000\,\tfrac{0.16^2}{4} = 0$$

$$u_D = 365.7\ ms^{-1}$$

4. (a) $\begin{bmatrix} 160 & 0 \\ 0 & 20 \end{bmatrix}\begin{bmatrix} \ddot{\theta} \\ \ddot{\gamma} \end{bmatrix} + \begin{bmatrix} 5V & 0 \\ -2V & V \end{bmatrix}\begin{bmatrix} \dot{\theta} \\ \dot{\gamma} \end{bmatrix} + \begin{bmatrix} k & 2V^2 \\ 0 & 4k-2V^2 \end{bmatrix}\begin{bmatrix} \theta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 160 & 0 \\ 0 & 20 \end{bmatrix}\begin{bmatrix} \ddot{\theta} \\ \ddot{\gamma} \end{bmatrix} + V\begin{bmatrix} 5 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} \dot{\theta} \\ \dot{\gamma} \end{bmatrix} + \left\{V^2\begin{bmatrix} 0 & 2 \\ 0 & -2 \end{bmatrix} + \begin{bmatrix} k & 0 \\ 0 & 4k \end{bmatrix}\right\}\begin{Bmatrix} \theta \\ \gamma \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$a_{11} = 160$, $a_{12} = 0$, $a_{21} = 0$, $a_{22} = 20$

$b_{11} = 5$, $b_{12} = 0$, $b_{21} = -2$, $b_{22} = 1$

$c_{11} = 0$, $c_{12} = 2$, $c_{21} = 0$, $c_{22} = -2$

$e_{11} = k = xV^2$, $e_{22} = 4k = \mu e_{11} = \mu xV^2 = \mu k$ $\quad \therefore \mu = 4$

$b_4 = (a_{11}a_{22} - a_{21}a_{12}) = 160(20) = 3200$

$b_3 = (a_{11}b_{22} + b_{11}a_{22} - a_{21}b_{12} - a_{12}b_{21})V = [(160)(1) + 5(20)]V = 260V$

$b_2 = [(\mu a_{11} + a_{22})x + (a_{11}c_{22} + b_{11}b_{22} + c_{11}a_{22} - a_{21}c_{12} - b_{12}b_{21} - c_{21}a_{12})]V^2$

$\quad = [(4\cdot160 + 20)x + (160\cdot -2 + 5\cdot1)]V^2 = (660x - 315)V^2$

$b_1 = [(\mu b_{11} + b_{22})x + (b_{11}c_{22} + c_{11}b_{22} - b_{21}c_{12} - b_{12}c_{21})]V^3$

$\quad = [(4\cdot5+1)x + (5\cdot -2 - (-2)(2))]V^3 = (21x - 6)V^3$

$b_0 = [\mu x^2 + (c_{22} + \mu c_{11})x + c_{11}c_{22} - c_{12}c_{21}]V^4$

$\quad = [4x^2 - 2x]V^4$

$b_4\lambda^4 + b_3\lambda^3 + b_2\lambda^2 + b_1\lambda + b_0 = 0$

$3200\lambda^4 + 260V\lambda^3 + (660x - 315)V^2\lambda^2 + (21x-6)V^3\lambda + (4x^2 - 2x)V^4 = 0$

At flutter speed, $\lambda = \pm j\omega_F$ and $\omega_F^2 = \frac{b_1}{b_3} = \frac{(21x-6)V^3}{260V} = \frac{(21x-6)V^2}{260}$

$$b_4\omega^4 - b_2\omega^2 + b_0 = 0$$

$$3200\left(\frac{21x-6}{260}\right)^2V^4 - (660x - 315)V^2\left(\frac{21x-6}{260}\right)V^2 + (4x^2 - 2x)V^4 = 0$$

$$3200\left(\frac{441x^2 - 252x + 36}{67600}\right) - \frac{693}{13}x^2 + \frac{2115}{52}x - \frac{189}{26} + 4x^2 - 2x = 0$$

$$-\frac{4805}{169}x^2 + \frac{18079}{676}x - \frac{1881}{338} = 0$$

$x_1 = 0.629894396$ $x_2 = 0.310740358$

$k = xV^2$

$\therefore V = \sqrt{\frac{k}{x}} = \sqrt{\frac{2.5 \times 10^4}{x}}$

$V_{F1} = 199.22\ ms^{-1}$ $V_{F2} = 283.64\ ms^{-1}$

(b) $w_{F1}/2\pi = \sqrt{\frac{(21x-6)V^2}{240}}/2\pi$

$w_{F1} = 5.29\ Hz$ $w_{F2} = 2.03\ Hz$

(c) The higher frequency is more meaningful as it corresponds to the lower flutter speed. An aircraft will increase its speed from low to high value. Once the system encounters flutter, it will undergo failure and wouldn't be able to increase its speed to V_{F2} anymore.

NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER 1 EXAMINATION 2014 – 2015

MA4704/AE4613 AEROELASTICITY

November/December 2014 Time Allowed : 2 ½ hours

INSTRUCTIONS

1. This paper contains **FOUR (4)** questions and comprises **SEVEN (7)** pages.
2. Answer **ALL** questions
3. Marks for each question are as indicated.
4. This is an **OPEN-BOOK** Examination.

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1. A simply- supported uniform beam of length $l = 1m$ and second moment of area $I = 1.0\times 10^{-6}\ m^4$ is shown in Figure 1(a). Young's modulus E is assumed to be $E = 2.1\times 10^{11}\ N/m^2$ and density $\rho A = 40kg/m$.

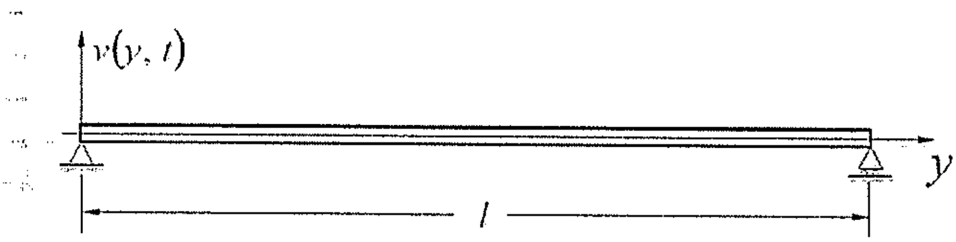

Figure 1(a): A simply-supported uniform beam

(a) Find its natural frequencies and associated mode shapes.

(5 marks)

Note: Question 1 continues on page 2.

(b) In the case of free vibration, following initial displacement is generated when a constant static moment M is applied at one of its ends as shown in Figure 1(b) while the initial velocity over the beam is assumed to be zero:

Figure 1(b): A simply-supported beam subjected to a constant moment

$$v(y,\,0) = f(y) = -0.01\,y\left(1-\frac{y^2}{l^2}\right) \qquad 0 \le y \le l$$
$$\dot{v}(y,\,0) = g(y) = 0 \qquad 0 \le y \le l$$

If the thus applied moment M is suddenly removed, find the subsequent free vibration of the beam.

(10 marks)

You may use the following integrations given:

$$\int_0^1 y\sin(i\pi y)\,dy = -\frac{1}{i\pi}\cos i\pi,\quad \int_0^1 y^3\sin(i\pi y)\,dy = -\left(\frac{1}{i\pi}+\frac{6}{i^3\pi^3}\right)\cos i\pi$$

(c) In the case of forced vibration, if a distributive force $q = 100\sin 100t$ is applied between $y = l/2$ and $y = l$, as shown in Figure 1(c), determine the bending vibration of the beam. (You are only required to find the particular solution assuming the homogeneous part will decay in practice).

(10 marks)

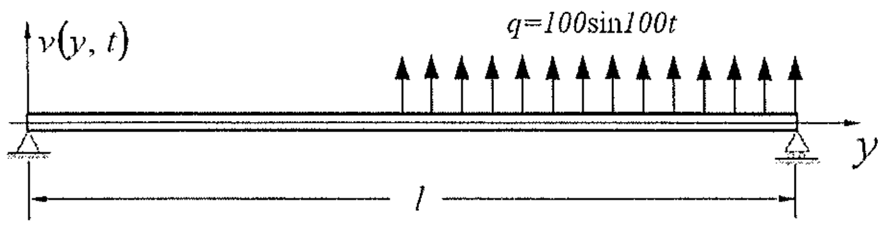

Figure 1(c): A simply-supported beam subjected to distributive sinusoidal force

Note: Question 1 continues on page 3.

(d) By examining the known mode shapes of the simply-supported beam, derive the natural frequencies and the mode shapes of the same beam but with 3 simple supports as shown in Figure 1(d).

(5 marks)

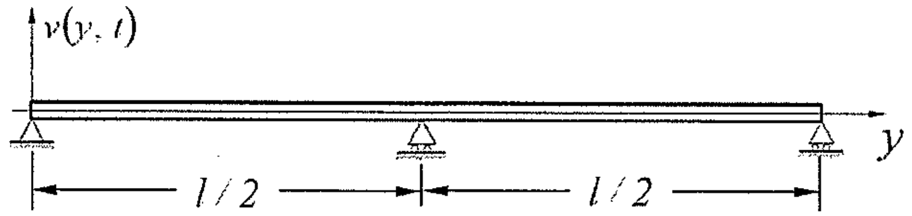

Figure 1(d): A beam with 3 simple supports

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2. A rigid model wing with aileron is elastically supported in a wind tunnel, as shown in Figure 2. Geometric parameters of the wing are: $S = 0.8m^2$, $c = 0.4m$, $e = 0.1m$, $a = 0.18m$. Aerodynamic coefficients are: $C_{L\alpha} = 6\ rad^{-1}$, $C_{M0} = 0.0\ rad^{-1}$, $C_{L\beta} = 1.0\ rad^{-1}$ and $C_{M\beta} = -0.6\ rad^{-1}$. Elastic torsional deformations θ are measured during test at different airspeeds U and aileron deflections β. Given these measured data: $\theta_1 = 1^\circ$ when $U_1 = 100$ m/s and $\beta_1 = 2.5^\circ$, and $\theta_2 = 5^\circ$ when $U_2 = 200$ m/s and $\beta_2 = 8.5^\circ$. Air density ρ is assumed to be $\rho = 1.2\ kg/m^3$.

(a) Determine the torsional stiffness k_T and hence the divergence speed U_D and the reversal speed U_R of the system.

(13 marks)

(b) Find the aileron efficiency η when dynamic pressure $q = 6000$ N/m^2.

(6 marks)

(c) To further enhance the reversal speed, an additional spring k=80000N/m is attached at the flap hinge as shown in Figure 2(b), find the new reversal speed.

(6 marks)

Hint: You may use the following general formula for elastic torsional deformation θ,

$$\theta = \frac{qS[cC_{M0} + eC_{L\alpha}\alpha_r + (eC_{L\beta} + cC_{M\beta})\beta]}{k_T - qeSC_{L\alpha}}$$

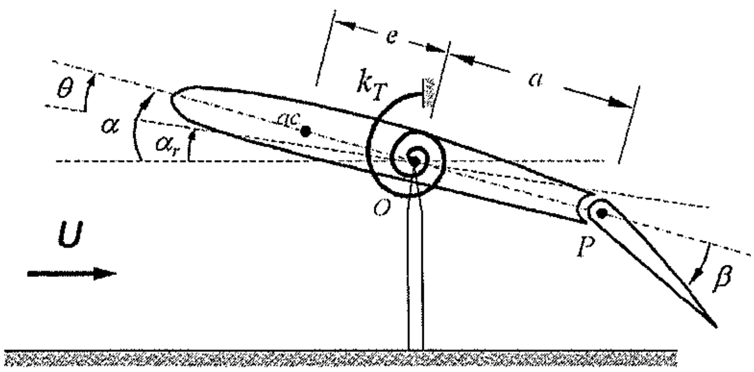

Figure 2(a) An elastically restrained wing with aileron

Note: Figure 2(b) appears on page 5.

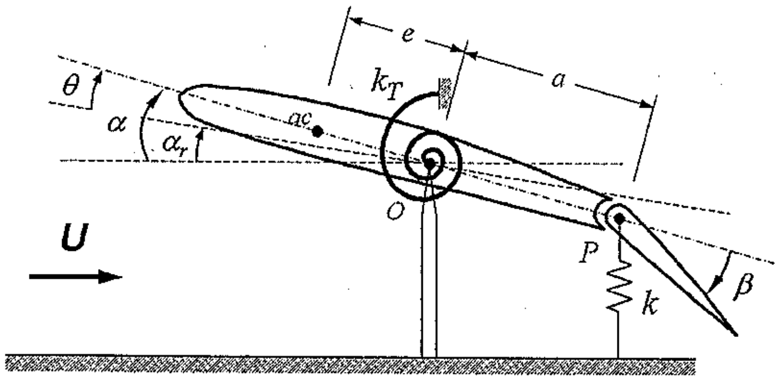

Figure 2(b): An elastically restrained wing with additional spring support

3. A typical airfoil section subjected to aerodynamic forces is shown in Figure 3, where m is the mass of the airfoil, I_A the mass moment of inertial about its elastic axis A, a the distance between the mass centre C and the elastic axis A, k and k_T are the translational and torsional spring constants, respectively.

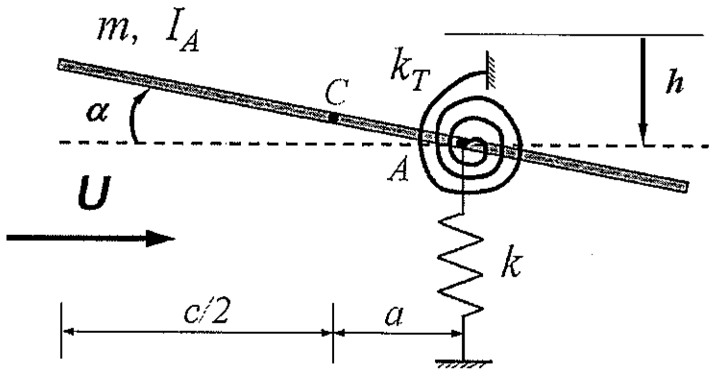

Figure 3: A typical airfoil section in an aerodynamic environment

Note: Question 3 continues on page 6.

For a given set of known system parameters and ignoring the effect of aerodynamic inertial and aerodynamic damping, following aeroelastic equations of motion, in inertial coupled form, in terms of non-dimensional plunge degree of freedom $\eta \equiv h/b$ (where $b = c/2$) and pitch degree of freedom α, can be established,

$$\begin{bmatrix} 100 & 20 \\ 20 & 25 \end{bmatrix} \begin{Bmatrix} \ddot{\eta} \\ \ddot{\alpha} \end{Bmatrix} + \begin{bmatrix} 3000 & 0 \\ 0 & 300000 \end{bmatrix} \begin{Bmatrix} \eta \\ \alpha \end{Bmatrix} = 2\,C(k)\,\overline{U}^2 \begin{bmatrix} 0 & -10 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \eta \\ \alpha \end{Bmatrix}$$

where the terms on the right hand are the simplified aerodynamic force and moment acting on the airfoil when ignoring the effect of aerodynamic inertial and aerodynamic damping, $\overline{U} \equiv U/b$ is the non-dimensional air speed and $C(k)$ is Theodorsen's function.

(a) Assume that Theodorsen's function $C(k) = 1$ and the oscillations of the airfoil are,

$$\begin{Bmatrix} \eta \\ \alpha \end{Bmatrix} = \begin{Bmatrix} H \\ A \end{Bmatrix} e^{i\omega t},$$

derive the characteristic equation of the system in terms of frequency ω and non-dimensional speed $\overline{U}$.

Hint: This can be established by substituting the assumed motion into the aeroelastic equations of motion and by letting the determinant be zero to enable non-trivial solutions to be found!

(10 marks)

(b) The characteristic equation has in general two roots (two natural frequencies of oscillation), each depends on the value of $\overline{U}$. Determine the lowest value of $\overline{U}$ when the two roots become equal.

(10 marks)

4. A binary aeroelastic system (e.g. wing bending-torsion system) is undergoing time-harmonic motion in terms of coordinates γ and θ as given as

$$\begin{bmatrix} \theta \\ \gamma \end{bmatrix} = \begin{bmatrix} \theta_0 \\ \gamma_0 \end{bmatrix} e^{j\omega t}$$

If the governing equations describing the binary aeroelatic system are given as

$$120\ddot{\theta} + 6V\dot{\theta} + k\theta + 4V^2\gamma = 0$$

$$10\ddot{\gamma} - 3V\dot{\theta} + V\dot{\gamma} + (3k - 3V^2)\gamma = 0$$

Then

(a) Determine the divergence speeds if k=90000.

(5 marks)

(b) Find the flutter speed that gives a flutter frequency of $\omega/2\pi$=10 Hz. If more than one flutter speeds are calculated, then identify which flutter speed is more physically meaningful and explain why.

(12 marks)

(c) Determine the flutter frequency corresponding to flutter speed of V=250 m/s. please briefly discuss the relationship between the flutter frequency and flutter speed.

(8 marks)

END OF PAPER

1. (a) $w_i = (\alpha_i l)^2 \sqrt{\frac{EI}{\rho A l^4}}$

$= (i\pi)^2 \sqrt{\frac{(2.1\times10^{11})(1\times10^{-6})}{40(1)^4}} = \sqrt{5250}\,\pi^2 i^2, \quad i=1,2,\dots$

$\phi_i(y) = \sin(\alpha_i y) = \sin\left(\frac{i\pi y}{l}\right) = \sin(i\pi y), \quad i=1,2,\dots$

(b) $v(y,t) = \sum_{i=1}^{\infty} \phi_i(y)(A_i \cos w_i t + B_i \sin w_i t)$

$\dot v(y,t) = \sum_{i=1}^{\infty} \phi_i(y)(-w_i A_i \sin w_i t + w_i B_i \cos w_i t)$

$v(y,0) = \sum_{i=1}^{\infty} \phi_i(y) A_i = -0.01\, y\left(1-\frac{y^2}{l^2}\right)$

Multiply both sides by $\phi_j(y)$ and integrate over $[0, l]$,

$\sum_{i=1}^{\infty} A_i \int_0^l \phi_i(y)\phi_j(y)\,dy = \int_0^l -0.01\,y\left(1-\frac{y^2}{l^2}\right)\phi_j(y)\,dy$

$\sum_{i=1}^{\infty} A_i \int_0^l \sin(i\pi y)\sin(j\pi y)\,dy = -0.01\left[\int_0^1 y\sin(i\pi y)\,dy - \frac{1}{l^2}\int_0^1 y^3 \sin(i\pi y)\,dy\right]$

$A_i \frac{l}{2} = -0.01\left[-\frac{(-1)^i}{i\pi} - \left(-\frac{(-1)^i}{i\pi} - \frac{6(-1)^i}{(i\pi)^3}\right)\right]$

$\frac{A_i}{2} = -0.06\frac{(-1)^i}{(i\pi)^3}$

$A_i = -0.12\frac{(-1)^i}{(i\pi)^3}$

$\dot v(y,0) = g(y) = 0$

$\sum_{i=1}^{\infty} \phi_i(y) w_i B_i = 0 \Rightarrow B_i = 0$

$\therefore v(y,t) = \sum_{i=1}^{\infty} \phi_i(y)\left[-0.12\frac{(-1)^i}{(i\pi)^3}\cos w_i t\right]$

$= \sum_{i=1}^{\infty} \sin(i\pi y)\left[-0.12\frac{(-1)^i}{(i\pi)^3}\cos \sqrt{5250}\,\pi^2 i^2 t\right]$

(c) $v(y,t) = \sum_{i=1}^{\infty}\phi_i(y)\xi_i(t)$

$\rho A\sigma_i \ddot\xi_i + \rho A \sigma_i w_i^2 \xi_i = \theta_i(t) \quad (i=1,2\dots) \qquad \sigma = \int_0^l \sin(i\pi y)\sin(j\pi y)\,dy = \frac{l}{2} = \frac{1}{2}$

$\theta_i = \int_0^l F(y,t)\phi_i(y)\,dy$

$= \int_{l/2}^{l} (100\sin 100t)(\sin(i\pi y))\,dy$

$= 100\sin 100t\left[-\frac{\cos(i\pi y)}{i\pi}\right]_{1/2}^{1}$

$= (100\sin 100t)\left[-\frac{(-1)^i}{i\pi}\right] = -\frac{(-1)^i}{i\pi}(100\sin 100t)$

$\ddot\xi_i + w_i^2\xi_i = \frac{1}{\rho A\sigma_i}\theta_i(t)$

$= \frac{1}{40(1/2)}\left[-\frac{(-1)^i}{i\pi}(100\sin 100t)\right]$

$= -\frac{(-1)^i}{i\pi}(5\sin 100t) \qquad w_i = \sqrt{5250}\,\pi^2 i^2$

let $\xi = A\sin 100t \Rightarrow \ddot\xi = -100^2 A\sin 100t$

$-100^2 A\sin 100t + A w_i^2 \sin 100t = -\frac{(-1)^i}{i\pi}(5\sin 100t)$

$A = -\frac{5(-1)^i}{i\pi(w_i^2-100^2)} = -\frac{5(-1)^i}{i\pi(5250\pi^4 i^4 - 100^2)}$

$\therefore \xi = -\frac{5(-1)^i}{i\pi(5250\pi^4 i^4-100^2)}\sin 100t$

$\therefore v(y,t) = \sum_{i=1}^{\infty}\sin(i\pi y)\left[-\frac{5(-1)^i}{i\pi(5250\pi^4 i^4 - 10000)}\sin 100t\right]$

$= \sum_{i=1}^{\infty}\sin(i\pi y)\left[-\frac{(-1)^i}{i\pi(1050\pi^4 i^4-2000)}\sin 100t\right]$

(d) With 2 simple supports, $w_{i,0} = \sqrt{40}\,\pi^2 i^2$, $\phi_{i,0}(y) = \sin\left(\frac{i\pi y}{1}\right) = \sin(i\pi y)$

$v(l/2, t) = 0 \Rightarrow X(l/2) = 0$

$M(l/2, t) = 0 \Rightarrow X''(l/2) = 0$

$X(l/2) = 0 \Rightarrow D_2 \sinh\left(\frac{\alpha l}{2}\right) + D_4 \sin\left(\frac{\alpha l}{2}\right) = 0$

$X''(l/2) = 0 \Rightarrow D_2 \sinh\left(\frac{\alpha l}{2}\right) - D_4 \sin\left(\frac{\alpha l}{2}\right) = 0$

$\begin{bmatrix}\sinh\left(\frac{\alpha l}{2}\right) & \sin\left(\frac{\alpha l}{2}\right)\\ \sinh\left(\frac{\alpha l}{2}\right) & -\sin\left(\frac{\alpha l}{2}\right)\end{bmatrix}\begin{bmatrix}D_2\\ D_4\end{bmatrix} = 0$

$\begin{vmatrix}\sinh\left(\frac{\alpha l}{2}\right) & \sin\left(\frac{\alpha l}{2}\right)\\ \sinh\left(\frac{\alpha l}{2}\right) & -\sin\left(\frac{\alpha l}{2}\right)\end{vmatrix} = 0$

$\Rightarrow -2\sinh\left(\frac{\alpha l}{2}\right)\sin\left(\frac{\alpha l}{2}\right) = 0 \Rightarrow \alpha_i = \frac{2i\pi}{l}$

$\therefore w_{i\,new} = (2i\pi)^2\sqrt{\frac{EI}{\rho A (l/2)^2}}$

$= 4i^2\pi^2\sqrt{\frac{(2.1\times10^{11})(1\times10^{-6})}{40\times(1/2)^2}}$

$= 4\sqrt{21000}\,i^2\pi^2$

$\phi_{i\,new} = \sin(\alpha_i y) = \sin\left(\frac{2i\pi y}{l}\right) = \sin(2i\pi y)$

2 $\theta = \dfrac{qS[cC_{m0} + eC_{L\alpha}\alpha_r + (eC_{L\beta} + cC_{m\beta})\beta]}{k_T - q_e S C_{L\alpha}}$

when $u_1 = 100$ & $\beta_1 = 2.5°$, $\theta_1 = 1°$

$$\frac{1\pi}{180°} = \frac{\frac{1}{2}(1.2)(100)^2[0.1[6\alpha_r) + (0.1 \cdot 1 + 0.4 \cdot -0.6)(\frac{2.5\times\pi}{180})](0.8)}{k_T - \frac{1}{2}(1.2)(100)^2(0.1)(0.8)(6)}$$

$$\frac{\pi}{180}k_T - 16\pi = 2880\,\alpha_r - \frac{28}{3}\pi$$

$$k_T = \frac{518400}{\pi}\alpha_r + 1200 \quad \text{—①}$$

When $u_2 = 200$ & $\beta_1 = 8.5°$, $\theta_2 = 5°$

$$\frac{5\pi}{180°} = \frac{\frac{1}{2}(1.2)(200)^2(0.8)[0.1(6\alpha_r) + (0.1 \cdot 1 + 0.4 \cdot -0.6)(\frac{8.5\times\pi}{180})]}{k_T - \frac{1}{2}(1.2)(200)^2(0.1)(0.8)(6)}$$

$$\frac{5\pi}{180}k_T - 320\pi = 11520\alpha_r - \frac{1904}{15}\pi$$

$$k_T = \frac{414720}{\pi}\alpha_r + 6950.4 \quad \text{—②}$$

①:②, $\alpha_r = 0.174$ rad $\quad k_T = 29952$ Nm/rad

$$q_D = \frac{k}{SC_{L\alpha}e} = \frac{29952}{0.8(6)(0.1)} = 62400\ N/m^2$$

$$u_D = \sqrt{\frac{2q_D}{\rho}} = \sqrt{\frac{2(62400)}{1.2}} = 322.5\ ms^{-1}$$

$$q_R = -\frac{kC_{L\beta}}{C_{L\alpha}ScC_{m\beta}} = -\frac{29952(1)}{6(0.8)(0.4)(-0.6)} = 26000\ N/m^2$$

$$u_R = \sqrt{\frac{2q_R}{\rho}} = \sqrt{\frac{2(26000)}{1.2}} = 208.2\ ms^{-1}$$

(b) $\eta = \dfrac{1 - q/q_R}{1 - q/q_D} = \dfrac{1 - 6000/26000}{1 - 6000/62400}$ ~~$= 0.851 = 85.1\%$~~

(c) $M_{ac} + Le = k_T\theta + k\theta a^2 = (k_T + ka^2)\theta$

Equivalent torsional stiffness $= k_T + ka^2 = 29952 + 80000(0.18)^2 = 32544$ N/m

$$q_R = -\frac{kC_{L\beta}}{C_{L\alpha}ScC_{m\beta}} = -\frac{32544(1)}{6(0.8)(0.4)(-0.6)} = 28250\ N/m^2$$

$$u_R = \sqrt{\frac{2q_R}{\rho}} = \sqrt{\frac{2(28250)}{1.2}} = 216.99\ ms^{-1}$$

3. $\eta = h/b$, $\quad b = c/2$, $\quad \bar{u} = u/b$

(a) $$\begin{bmatrix}100 & 20\\20 & 25\end{bmatrix}\begin{Bmatrix}\ddot{\eta}\\\ddot{\alpha}\end{Bmatrix} + \begin{bmatrix}3000 & 0\\0 & 300000\end{bmatrix}\begin{Bmatrix}\eta\\\alpha\end{Bmatrix} = 2\bar{u}^2\begin{bmatrix}0 & -10\\0 & 1\end{bmatrix}\begin{Bmatrix}\eta\\\alpha\end{Bmatrix}$$

$$\begin{Bmatrix}\eta\\\alpha\end{Bmatrix} = \begin{Bmatrix}H\\A\end{Bmatrix}e^{i\omega t} \qquad \begin{Bmatrix}\ddot{\eta}\\\ddot{\alpha}\end{Bmatrix} = -\omega^2\begin{Bmatrix}H\\A\end{Bmatrix}e^{i\omega t}$$

$$-\omega^2\begin{bmatrix}100 & 20\\20 & 25\end{bmatrix}\begin{Bmatrix}H\\A\end{Bmatrix}e^{i\omega t} + \begin{bmatrix}3000 & 0\\0 & 300000\end{bmatrix}\begin{Bmatrix}H\\A\end{Bmatrix}e^{i\omega t} = 2\bar{u}^2\begin{bmatrix}0 & -10\\0 & 1\end{bmatrix}\begin{Bmatrix}H\\A\end{Bmatrix}e^{i\omega t}$$

$$\begin{bmatrix}3000 - 100\omega^2 & 20\bar{u}^2 - 20\omega^2\\-20\omega^2 & 300000 - 2\bar{u}^2 - 25\omega^2\end{bmatrix}\begin{Bmatrix}H\\A\end{Bmatrix} = \begin{Bmatrix}0\\0\end{Bmatrix}$$

To enable non-trivial/non-zero solutions to be found, the determinant has to vanish.

$$\begin{vmatrix}3000 - 100\omega^2 & 20\bar{u} - 20\omega^2\\-20\omega^2 & 300000 - 2\bar{u}^2 - 25\omega^2\end{vmatrix} = 0$$

$$(3000 - 100\omega^2)(300000 - 2\bar{u}^2 - 25\omega^2) - (-20\omega^2)(20\bar{u}^2 - 20\omega^2) = 0$$

$$(30 - \omega^2)(300000 - 0.02\bar{u}^2 - 0.25\omega^2) - (-0.2\omega^2)(0.2\bar{u}^2 - 0.2\omega^2) = 0$$

$$90000 - 0.6\bar{u}^2 - 7.5\omega^2 - 3000\omega^2 + 0.02\omega^2\bar{u}^2 + 0.25\omega^4 + 0.04\omega^2\bar{u}^2 - 0.04\omega^4 = 0$$

$$0.21\omega^4 + (-3007.5 + 0.06\bar{u}^2)\omega^2 + (90000 - 0.6\bar{u}^2) = 0$$

(b) $b^2 - 4ac = 0$

$(-3007.5 + 0.06\bar{u}^2)^2 - 4(0.21)(90000 - 0.6\bar{u}^2) = 0$

$9045056.25 - 360.9\bar{u}^2 + (3.6\times10^{-3})\bar{u}^4 - 75600 + 0.504\bar{u}^2 = 0$

$(3.6\times10^{-3})\bar{u}^4 - 360.396\bar{u}^2 + 8969456.25 = 0$

$\bar{u}_1^2 = 46315.02674$ $\qquad \bar{u}_2^2 = 53794.97326$

$\bar{u}_1 = 215.21/s$

4. $\begin{bmatrix}120 & 0\\ 0 & 10\end{bmatrix}\begin{Bmatrix}\ddot{\theta}\\ \ddot{\gamma}\end{Bmatrix} + \begin{bmatrix}6V & 0\\ -3V & V\end{bmatrix}\begin{Bmatrix}\dot{\theta}\\ \dot{\gamma}\end{Bmatrix} + \begin{bmatrix}k & 4V^2\\ 0 & 3k-3V^2\end{bmatrix}\begin{Bmatrix}\theta\\ \gamma\end{Bmatrix} = \begin{Bmatrix}0\\ 0\end{Bmatrix}$

(a) $\begin{bmatrix}120 & 0\\ 0 & 10\end{bmatrix}\begin{Bmatrix}\ddot{\theta}\\ \ddot{\gamma}\end{Bmatrix} + \begin{bmatrix}6V & 0\\ -3V & V\end{bmatrix}\begin{Bmatrix}\dot{\theta}\\ \dot{\gamma}\end{Bmatrix} + \begin{bmatrix}90000 & 4V^2\\ 0 & 270000-3V^2\end{bmatrix}\begin{Bmatrix}\theta\\ \gamma\end{Bmatrix} = \begin{Bmatrix}0\\ 0\end{Bmatrix}$

for divergence speed, consider only the displacement related terms.

$$\begin{bmatrix}90000 & 4V^2\\ 0 & 270000-3V^2\end{bmatrix}\begin{Bmatrix}\theta\\ \gamma\end{Bmatrix} = \begin{Bmatrix}0\\ 0\end{Bmatrix}$$

$$\begin{vmatrix}90000 & 4V^2\\ 0 & 270000-3V^2\end{vmatrix} = \begin{Bmatrix}0\\ 0\end{Bmatrix}$$

$90000(270000 - 3V^2) = 0$ $\qquad \Rightarrow V = 300\ ms^{-1}$

(b) $a_{11} = 120,\quad a_{12} = 0,\quad a_{21} = 0,\quad a_{22} = 10$

$b_{11} = 6,\quad b_{12} = 0,\quad b_{21} = -3,\quad b_{22} = 1$

$c_{11} = 0,\quad c_{12} = 4,\quad c_{21} = 0,\quad b_{22} = -3$

$e_{11} = k = xV^2,\quad e_{22} = 3k = \mu e_{11} \quad \Rightarrow \mu = 3$

$b_4 = (a_{11}a_{22} - a_{21}a_{12}) = 1200$

$b_3 = (a_{11}b_{22} + b_{11}a_{22} - a_{21}b_{12} - a_{12}b_{21})V = (120\cdot1 + 6\cdot10)V = 180V$

$b_2 = [(\mu a_{11} + a_{22})x + (a_{11}c_{22} + b_{11}b_{22} + c_{11}a_{22} - a_{21}c_{12} - b_{12}b_{21} - c_{21}a_{12})]V^2$

$= [(360+10)x + (-360+6)]V^2 = (370x - 354)V^2$

$b_1 = [(\mu b_{11} + b_{22})x + (b_{11}c_{22} + c_{11}b_{22} - b_{21}c_{12} - b_{12}c_{21})]V^3$

$= [(18+1)x + (-18+12)]V^3 = (19x - 6)V^3$

$b_0 = [\mu x^2 + (c_{22} + \mu c_{11})x + c_{11}c_{22} - c_{12}c_{21}]V^4$

$= [3x^2 + (-3)x + 0]V^4 = (3x^2 - 3x)V^4$

$b_4\lambda^4 + b_3\lambda^3 + b_2\lambda^2 + b_1\lambda + b_0 = 0$

$1200\lambda^4 + 180V\lambda^3 + (370x - 354)V^2\lambda^2 + (19x-6)V^3\lambda + (3x^2 - 3x)V^4 = 0$

$w_F^2 = \frac{b_1}{b_3} = \frac{(19x-6)V^3}{180V} = \frac{(19x-6)V^2}{180}$

$b_4w^4 - b_2w^2 + b_0 = 0$

$1200\left(\frac{(19x-6)V^2}{180}\right)^2 - (370x - 354)V^2\left(\frac{(19x-6)V^2}{180}\right) + (3x^2 - 3x)V^4 = 0$

$\frac{1}{27}(36x^2 - 228x + 36) - \frac{1}{180}(7030x^2 - 8946x + 2124) + 3x^2 - 3x = 0$

$-\frac{625}{18}x^2 + \frac{3443}{90}x - \frac{157}{15} = 0$

$x_1 = 0.595921918$ $\qquad x_2 = 0.505838081$

$w_F = 20x,\quad (20x)^2 = \frac{(19(0.596) - 6)V^2}{180}$

$V_1 = 365.3\ ms^{-1}$

$x = x_2,\quad V_2 = 443.6\ ms^{-1}$

$\therefore x = x_1$ more meaningful as flutter will happen at lower speed.

(c) $w_F = \sqrt{\frac{(19x-6)(250)^2}{180}}$

$x = x_1$, $w_F = 42.99 \text{ rads}^{-1} = 6.84 \text{ Hz}$

$x = x_2$, $w_F = 35.41 \text{ rads}^{-1} = 5.64 \text{ Hz}$

When flutter speed increases, flutter frequency will increase provided the x is kept constant.