NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER 1 EXAMINATIONS 2012-2013

MH1811 - MATHEMATICS 2

November / December 2012 Time Allowed: 2½ hours

INSTRUCTIONS

1. This paper contains 4 questions and comprises 6 pages.
2. Answer ALL questions.
3. All questions carry equal marks.
4. This is a closed-book examination.
5. A list of formulae is given in Appendix A (Pages 4-6).

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1. (a) Use the two-path test to show that the limit below does not exist.

$$\lim_{x\to 0, y\to 0}\left(\frac{x^3+9x^2y+9xy^2+3y^3}{x^3-9x^2y+9xy^2-3y^3}\right)$$

(5 Marks)

(b) u is a function of v and w, where $v = x + 2t$ and $w = x - 2t$. Express $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial t}$ in terms of $\frac{\partial u}{\partial v}$ and $\frac{\partial u}{\partial w}$. If $\frac{\partial^2 u}{\partial v \partial w} = 0$, show that $\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2}$.

(7 Marks)

(c) The temperature distribution on a flat plate is given as $T(x,y) = 50e^{-(x^2+y^2)/20}$. An ant located at (2, 1) of the plate wants to move to a cooler location. What direction should it move towards so that the decrease in temperature is the fastest?

(5 Marks)

(d) A hollow sphere has a radius of 3. A solid cylinder is placed inside the sphere. Determine the length and the base area (circular in shape) of the cylinder that will result in having the smallest unoccupied space inside the sphere. Hence, determine the volume of the unoccupied space. [Note: the volume of a sphere is $\frac{4}{3}\pi r^3$ where r is the radius of the sphere.]

(8 Marks)

2. (a) Determine if the series $\sum_{n=1}^{\infty}\left(\frac{3n^2+2}{\sqrt{n^5}-5}\right)$ converges or diverges.

(6 Marks)

(b) Find the radius of convergence and the interval of convergence of the series $\sum_{n=0}^{\infty}\left[\frac{(-5)^n x^n}{\sqrt{n+2}}\right]$.

(9 Marks)

(c) Generate the Taylor's series (showing only the first 4 terms) of $\sqrt{x}$ at $a = 4$. If the corresponding Taylor's polynomial of degree 2 is used to approximate the value of $\sqrt{x}$ for $3 < x < 5$, use Taylor's theorem to estimate the accuracy of the approximation.

(10 Marks)

3. (a) The intersection of the surface $z = 4 - x^2 - y^2$ and the *x-y* plane (i.e. $z = 0$) is a circle. Determine the equation of this circle. Hence find the volume bounded by the surface and the region above the *x-y* plane (i.e. $z \geq 0$).

(11 Marks)

(b) Solve the first order ordinary differential equation:

$$(2y^2 + x)\frac{dy}{dx} = 2xe^x - y + 6x^2, \qquad y(1) = 1$$

(14 Marks)

4. (a) $(1-x)y''+xy'-y=2(x-1)^3e^x, \qquad 0<x<1$

(i) Verify e^x and x are solutions of the homogeneous equation corresponding to the above equation.

(ii) Find the general solution using the variations of parameters method. [Note: Use the result of part (i)].

(13 Marks)

(b) Using the D-operator method, solve

$$\frac{d^2y}{dx^2}+6\frac{dy}{dx}+9y=16e^{-3x}(\sin^2 x+\cos x)$$

(12 Marks)

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(1) (a) path 1 : $y = 0$

$$\lim_{x\to 0}\left(\frac{x^3}{x^3}\right) = 1$$

path 2 : $x = 0$

$$\lim_{y\to 0}\left(\frac{0+0+0+3y^3}{0+0+0-3y^3}\right) = \lim_{y\to 0}\left(\frac{3y^3}{-3y^3}\right) = -1$$

∴ because path 1 ≠ path 2, hence the limit does not exist.

(b) $u = f(v,w)$, $v = x+2t$, $w = x-2t$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial u}{\partial w}\cdot\frac{\partial w}{\partial x}$$

$$= \frac{\partial u}{\partial v}\cdot(1) + \frac{\partial u}{\partial w}(1) = \frac{\partial u}{\partial v} + \frac{\partial u}{\partial w}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial v}\cdot\frac{\partial v}{\partial t} + \frac{\partial u}{\partial w}\cdot\frac{\partial w}{\partial t}$$

$$= \frac{\partial u}{\partial v}(2) + \frac{\partial u}{\partial w}\cdot(2) = 2\left(\frac{\partial u}{\partial v} + \frac{\partial u}{\partial w}\right)$$

$$\rightarrow \frac{\partial^2 u}{\partial v\partial w} = 0$$

$$\frac{\partial(\frac{\partial u}{\partial x})}{\partial x} = \frac{\partial^2 u}{\partial x^2} = \frac{\partial(\frac{\partial u}{\partial x})}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial(\frac{\partial u}{\partial x})}{\partial w}\cdot\frac{\partial w}{\partial x}$$

$$= \left(\frac{\partial^2 u}{\partial v^2} + \frac{\partial^2 u}{\partial v\partial w}\right)\cdot(1) + \left(\frac{\partial^2 u}{\partial^2 w} + \frac{\partial^2 u}{\partial v\partial w}\right)\cdot(1) = \frac{\partial^2 u}{\partial v^2} + \frac{\partial^2 u}{\partial w^2}$$

$$\frac{\partial(\frac{\partial u}{\partial t})}{\partial t} = \frac{\partial^2 u}{\partial t^2} = \frac{\partial(\frac{\partial u}{\partial t})}{\partial v}\cdot\frac{\partial v}{\partial t} + \frac{\partial(\frac{\partial u}{\partial t})}{\partial v}\cdot\frac{\partial v}{\partial t}$$

$$= 2\left(\frac{\partial^2 u}{\partial v^2} + \frac{\partial^2 u}{\partial v\partial w}\right)\cdot(2) + 2\left(\frac{\partial^2 u}{\partial w^2} + \frac{\partial^2 u}{\partial v\partial w}\right)(2) = 4\left(\frac{\partial^2 u}{\partial v^2} + \frac{\partial^2 u}{\partial w^2}\right)$$

$$\therefore \frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2}$$

(c) $T(x,y) = 50\, e^{-(x^2+y^2)/20}$

$A(x,y) = (2,1)$

$$\nabla T(x,y) = \frac{\partial T}{\partial x}\hat{i} + \frac{\partial T}{\partial y}\hat{j} = \left(-5x\, e^{-(x^2+y^2)/20}\right)\hat{i} + \left(-5y\, e^{-(x^2+y^2)/20}\right)\hat{j}$$

$$\nabla T(2,1) = \left(-10 e^{-1/4}\right)\hat{i} + \left(-5 e^{-1/4}\right)\hat{j}$$

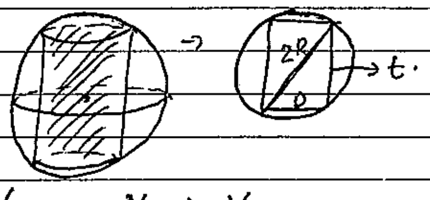
$D_u f = \nabla T\cdot\vec{u} = |\nabla T|\cdot|\vec{u}|\cdot\cos\theta = |\nabla T|\cos\theta$ ($\vec{u}$ = unit vector).

- for $D_u f$ to be minimal (the ant want to move to cooler location), $\cos\theta = -1$ or $\theta = 180°$. In other words:

$$\vec{u} = -\frac{\nabla T}{|\nabla T|}$$

$$\therefore \vec{u} = \left(\frac{+10 e^{-1/4}}{\sqrt{125}\, e^{-1/4}}\right)\hat{i} + \left(\frac{5 e^{-1/4}}{\sqrt{125}\, e^{-1/4}}\right)\hat{j} = \left(\tfrac{2}{5}\sqrt{5}\right)\hat{i} + \left(\tfrac{1}{5}\sqrt{5}\right)\hat{j}$$

(d)



$(2R)^2 = D^2 + t^2$

$D^2 = 4R^2 - t^2.$

$$V_{space} = V_{ball} - V_{cylinder} = \frac{4}{3}\pi R^3 - \frac{\pi}{4}D^2 t$$

$$V_{space} = \frac{4}{3}\pi R^3 - \frac{\pi}{4}(4R^2 - t^2)\, t$$

$$V_{space} = \frac{4}{3}\pi R^3 - \pi R^2 t + \frac{\pi}{4}t^3$$

local max / min point : $\frac{\partial V_{space}}{\partial t} = 0.$

$$0 - \pi R^2 + \frac{3}{4}\pi t^2 = 0 \rightarrow t = R\sqrt{\frac{4}{3}}$$

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$V_{max/min} = \frac{4}{3}\pi R^3 - \pi R^2 (R\sqrt{\frac{4}{3}}) + \frac{\pi}{4}\cdot\frac{4}{3}\cdot\sqrt{\frac{4}{3}} R^3$

$= \pi R^3 (\frac{4}{3} - \frac{2}{3}\sqrt{\frac{4}{3}}) = \frac{4}{3}\pi R^3 (1 - \frac{1}{3}\sqrt{3}) = V_{ball}(1-\frac{1}{3}\sqrt{3})$.

check : take $t = R$, hence $V' = \frac{4}{3}\pi R^3 - \pi R^3 + \frac{\pi}{4} R^3$

$V' = \frac{7}{12}\pi R^3 = \frac{7}{16} V_{ball} > V_{max/min}$

$\therefore$ Hence, this $V_{max/min}$ is V_{min}, not V_{max}. Thus, $V_{min} = \frac{4}{3}\pi R^3 (1 - \frac{1}{3}\sqrt{3})$.

(2) (a) $\Sigma a_n = \Sigma \left(\frac{3n^2+2}{\sqrt{n^5-5}}\right)$

take $\Sigma b_n = \Sigma \frac{1}{n}$, then Σb_n is diverges.

by limit comparison test:

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{3n^3+2n}{\sqrt{n^5-5}}\right) = \lim_{n\to\infty} \left(\frac{3\sqrt{n} + 2n^{-3/2}}{\sqrt{1-5n^{-5}}}\right) = \infty$

by limit comparison theorem, because Σb_n is diverges, then Σa_n is diverges too.

$\therefore$ Hence $\Sigma \left(\frac{3n^2+2}{\sqrt{n^5-5}}\right)$ is diverges.

(b) by ratio test:

$\lim_{n\to\infty} |r_n| = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$

$= \lim_{n\to\infty} \left| \frac{(-5)^{n+1} x^{n+1}}{\sqrt{n+3}} \cdot \frac{\sqrt{n+2}}{(-5)^n x^n} \right| < 1$

$= \lim_{n\to\infty} \left| -5x \cdot \frac{\sqrt{n+2}}{\sqrt{n+3}} \right| < 1$

$= |-5x| < 1$

$-1 < -5x < 1 \rightarrow -\frac{1}{5} < x < \frac{1}{5}$

$\therefore$ Thus, ROC $= \frac{1}{5}$.

check : $x = \frac{1}{5} \rightarrow \Sigma \left(\frac{(-1)^n}{\sqrt{n+2}}\right)$ = converges.

$x = -\frac{1}{5} \rightarrow \Sigma \left(\frac{1}{\sqrt{n+2}}\right)$ = diverges.

$\therefore$ Hence, IOC $= (-\frac{1}{5}, \frac{1}{5}]$

(c) $f(x) = \sqrt{x} = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$

at $a = 4$:

$\therefore \sqrt{x} = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3$

if degree 2 is used as approximation $\sqrt{x}$, then the accuracy is $\frac{1}{512}(x-4)^3$.

if $3 < x < 5$, then :

$\therefore -\frac{1}{512} \leq$ accuracy $\leq \frac{1}{512}$

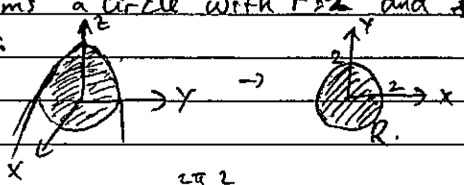
③ (a) $z = 4 - x^2 - y^2$

at $z=0 \rightarrow 0 = 4 - x^2 - y^2$

$x^2 + y^2 = 4 = 2^2$

it forms a circle with $r=2$ and have centre point at origin point.

The sketch :



$$V = \iint_R z\,dA = \int_0^{2\pi}\int_0^2 (4-r^2)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi} \left(2r^2 - \tfrac{1}{4}r^4\right)\Big|_0^2\,d\theta$$

$$\therefore = 4\,\theta\Big|_0^{2\pi} = 8\pi.$$

(b) $(2y^2+x)\frac{dy}{dx} = 2xe^x - y + 6x^2$, $y(1)=1$

$(2y^2+x)\,dy + (y - 2xe^x - 6x^2)\,dx = 0$

$N(x,y)\,dy + M(x,y)\,dx = 0$ → partial differentiation technique.

$\frac{\partial M}{\partial y} = 1$, $\frac{\partial N}{\partial x} = 1$.

because $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then it is not necessary to put integration factor (I).

$\frac{\partial u}{\partial x} = (y - 2xe^x - 6x^2) \rightarrow u(x,y) = xy - 2xe^x + 2e^x - 2x^3 + f(y)$

$\frac{\partial u}{\partial y} = x + f'(y) = x + 2y^2$

$f(y) = \frac{2}{3}y^3 + C.$

$\Rightarrow u(x,y) = 0.$

$xy - 2xe^x + 2e^x - 2x^3 + \frac{2}{3}y^3 + C = 0.$

$y(1)=1 \rightarrow 1.1 - 2.1e + 2e - 2 + \frac{2}{3} + C = 0 \rightarrow C = \frac{1}{3}$

$\therefore$ Hence: $xy - 2xe^x + 2e^x - 2x^3 + \frac{2}{3}y^3 = -\frac{1}{3}$

④ (a) (i) $(1-x)y'' + xy' - y = 2(x-1)^3 e^x$, $0 < x < 1$.

homogeneous equation :

$(1-x)y'' + xy' - y = 0.$

· for $y = e^x \rightarrow y' = e^x \rightarrow y'' = e^x$

$(1-x)y'' + xy' - y = (1-x)(e^x) + x(e^x) - e^x = 0$ (correct)

· for $y = x \rightarrow y' = 1 \rightarrow y'' = 0$

$(1-x)y'' + xy' - y = 0 + x - x = 0$ (correct).

$\therefore$ Hence, it is verified e^x and x are solutions of homogeneous equation.

(ii) $y'' + \frac{x}{1-x}y' - \frac{1}{1-x}y = -2(x-1)^2 e^x$

by various parameter method: $y_1 = x$, $y_2 = e^x$

$y_1 C_1' + y_2 C_2' = 0$

$y_1' C_1' + y_2' C_2' = -2(x-1)^2 e^x$

DISCLAIMER: *The solutions are done by students who scored A or above in this subject. The MAE Club and Campus supplies are not liable or responsible for any errors in the contents of these solutions. Students are advised to take the solutions as a guide rather than absolute answers to exam paper.*

or in matrix:

$$\begin{bmatrix} x & e^x \\ 1 & e^x \end{bmatrix}\begin{bmatrix} C_1' \\ C_2' \end{bmatrix} = \begin{bmatrix} 0 \\ -2(x-1)^2 e^x \end{bmatrix}$$

$$\begin{bmatrix} C_1' \\ C_2' \end{bmatrix} = \frac{1}{e^x(x-1)}\begin{bmatrix} e^x & e^x \\ -1 & x \end{bmatrix}\begin{bmatrix} 0 \\ -2(x-1)^2 e^x \end{bmatrix}$$

$$\begin{bmatrix} C_1' \\ C_2' \end{bmatrix} = \begin{bmatrix} 2(x-1)e^x \\ -2(x-1)x \end{bmatrix}$$

Thus: $C_1 = 2\int (x-1)e^x dx = 2(x-2)e^x + C_1$

$C_2 = -2\int (x-1)x\,dx = -\frac{2}{3}x^3 + x^2 + C_2$

$\therefore$ the solution: $y_p = C_1 y_1 + C_2 y_2$

$y_p = (2(x-2)e^x + C_1)x + (e^x(-\frac{2}{3}x^3 + x^2 + C_2))$

$y_p = C_1 x + C_2 e^x + e^x(-\frac{2}{3}x^3 + 3x^2 - 4x)$

(b) $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 16e^{-3x}(\sin^2 x + \cos x)$

$(D^2 + 6D + 9I)y = 16e^{-3x}(\sin^2 x + \cos x)$

$y = \frac{1}{D^2+6D+9I}(16e^{-3x}(\sin^2 x + \cos x))$

$y_p = \frac{1}{(D+3I)(D+3I)}(16e^{-3x}(\sin^2 x + \cos x))$

$y_p = \frac{1}{(D+3I)}(\frac{16}{3}e^{-3x}(\frac{3}{2}x + \frac{1}{2} - \frac{3}{4}\sin 2x + 3\sin x))$

$\therefore\ y_p = \frac{16}{9}e^{-3x}(\frac{3}{2}x + \frac{1}{2} + \frac{9}{4}x^2 + \frac{9}{8}\cos 2x - 9\cos x)$

NANYANG TECHNOLOGICAL UNIVERSITY
SEMESTER 2 EXAMINATION 2014–2015
MH1811 – MATHEMATICS 2

APRIL 2015 TIME ALLOWED: 2 HOURS

Matriculation Number:

Seat Number:

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains EIGHT (8) questions and comprises SIXTEEN (16) pages, including an Appendix.
2. Answer ALL questions. The marks for each question are indicated at the beginning of each question.
3. This IS NOT an OPEN BOOK exam. However, a list of formulae is provided in the Appendix.
4. Candidates may use calculators. However, they should write down systematically the steps in the workings.
5. All your solutions should be written in this booklet within the space provided after each question. If you use an additional answer book, attach it to this booklet and hand them in at the end of the examination.

For examiners only

Question	Marks
1 (15)	
2 (15)	
3 (10)	
4 (10)	

Question	Marks
5 (14)	
6 (12)	
7 (14)	
8 (10)	

TOTAL (100)

QUESTION 1. (15 Marks)

Consider the differential equation

$$x^2y'' + 3xy' + y = 4\ln x. - - - (*)$$

(a) Suppose $y_1(x)$ is a solution of the associated homogeneous differential equation of (*), and $y(x) = C(x)y_1(x)$ is a general solution of (*). Show that $C(x)$ satisfies the following differential equation

$$(2x^2y_1' + 3xy_1)\,C' + x^2y_1C'' = 4\ln x.$$

(b) Verify that $y_1(x) = x^{-1}$ is a solution of the associated homogeneous differential equation of (*).

Question 1 continues on page 3.

(c) Use parts (a) and (b) to solve the differential equation (*).

End of Question 1.

QUESTION 2. (15 Marks)

(a) Consider the series $\displaystyle\sum_{n=2}^{\infty}\left(\frac{1}{\sqrt[3]{3n+2}}-\frac{1}{\sqrt[3]{3n+5}}\right)$.

(i) Find a formula for the partial sum $s_m=\displaystyle\sum_{n=2}^{m}\left(\frac{1}{\sqrt[3]{3n+2}}-\frac{1}{\sqrt[3]{3n+5}}\right)$.

(ii) Is the series $\displaystyle\sum_{n=2}^{\infty}\left(\frac{1}{\sqrt[3]{3n+2}}-\frac{1}{\sqrt[3]{3n+5}}\right)$ convergent? Justify your answer.

Question 2 continues on page 5.

(b) Is the series $\sum_{n=3}^{\infty} \frac{\sin\left(\frac{1}{3\ln n}\right)}{n}$ is convergent? Justify your answer.

End of Question 2.

QUESTION 3. (10 Marks)

A tank with a capacity of 400L is full of a mixture of water and chlorine with a concentration of 0.05g of chlorine per liter. In order to reduce the concentration of chlorine, fresh water is pumped into the tank at a rate of 5L/s. The mixture is kept stirred and is pumped out at a rate of 10 L/s.

Determine the time t_0 when there is no mixture in the tank and find the amount of chlorine in the tank at time $t = \frac{1}{2}t_0$.

End of Question 3.

QUESTION 4. (10 Marks)

Verify that the following differential equation is exact and proceed to solve it.

$$\left[\sin(xy) + xy\cos(xy) + y^2 + e^{3x}\right]\ dx + \left[x^2\cos(xy) + 2xy\right]\ dy = 0.$$

End of Question 4.

QUESTION 5. (14 Marks)

Let $f(x) = \sqrt{4+x}$.

(a) Find the Taylor polynomial, $T_3(x)$, of $f(x) = \sqrt{4+x}$ at $a = 0$.

(b) Approximate $\sqrt{3.6}$ using your result in part (a).

Question 5 continues on page 9.

(c) The value $f(x) = \sqrt{4+x}$ is estimated by $T_2(x)$ for $|x| < 0.2$. Use the Taylor's remainder to show that the error is at most 2×10^{-4}.

End of Question 5.

QUESTION 6. (12 Marks)

Let $f(x,y) = \dfrac{1}{x^2 y} + \sqrt{xy}$.

(a) Determine the domain D of f and sketch the domain on an xy-plane.

(b) Determine the rate of change in f at $(1,4)$ along the direction $\mathbf{v} = \mathbf{i} - \mathbf{j}$.

Question 6 continues on page 11.

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(c) Suppose $x = e^{s+2t}$ and $y = t^2 + s^2$. Use Chain Rule to find $\dfrac{\partial f}{\partial t}$ when $s = -2$ and $t = 1$.

End of Question 6.

(14 Marks)

QUESTION 7.
Consider the function

$$f(x,y) = \begin{cases} x + \sqrt{y} & \text{if } y \geq 0; \\ \dfrac{x^6 - 4x^2y^2}{x^4y + 3y^3} & \text{if } y < 0. \end{cases}$$

(a) Does the limit $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exist? Justify your answer.

(b) Does $f_y(0,0)$ exist? If it exists, what is its value? Justify your answer.

Question 7 continues on page 13.

(c) Evaluate $\iint_R f(x,y)\,dx\,dy$ over the triangular region R with vertices $(0,0)$, $(0,2)$ and $(1,0)$.

End of Question 7.

QUESTION 8. (10 Marks)

Let $f(x,y) = (2x-x^2)(y^2-4y)$ and $S = \{(x,y) : x^2+y^2 > 4\}$. Classify all stationary points of f in the region S as local maximum, local minimum or saddle points.

End of Question 8.

END OF PAPER

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1. (a) $\because \quad x^2y'' + 3xy' + y = 4\ln x \quad -(*)$

Let $x^2y'' + 3xy' + y = 0$

$y(x) = C(x)\, y_1(x)$

$y' = C'y_1 + Cy_1'$

$y'' = C''y_1 + 2C'y_1' + Cy_1''$

$x^2y'' + 3xy' + y$

$= x^2(C''y_1 + 2C'y_1' + Cy_1'') + 3x(C'y_1 + Cy_1') + Cy_1$

$= C(x^2y_1'' + 3xy_1' + y_1) + C''x^2y_1 + 2C'x^2y_1' + 3C'xy_1$

$\because$ $y_1(x)$ is a solution of the associated homogenous differential equation of (*)

$\therefore \quad x^2y_1'' + 3xy_1' + y_1 = 0$

$x^2y'' + 3xy' + y$

$= C''x^2y_1 + 2C'x^2y_1' + 3C'xy_1$

$C''x^2y_1 + 2C'x^2y_1' + 3C'xy_1 = 4\ln x$

$(2x^2y_1' + 3xy_1)C' + x^2y_1C'' = 4\ln x$ (shown).

(b) Let $x^2y'' + 3xy' + y = 0$

$\because \quad y_1 = x^{-1}$

$y_1' = -x^{-2}$

$y_1'' = 2x^{-3}$

$\therefore \quad x^2(2x^{-3}) + 3x(-x^{-2}) + x^{-1} = 2x^{-1} - 3x^{-1} + x^{-1}$

$= 0$ (verified).

$\therefore$ $y_1(x) = x^{-1}$ is a solution of the associated homogeneous differential equation of (*).

(c) $(2x^2y_1' + 3xy_1)C' + x^2y_1C'' = 4\ln x$

$[2x^2(-x^{-2}) + 3x(x^{-1})]C' + x^2(x^{-1})C'' = 4\ln x$

$(-2+3)C' + xC'' = 4\ln x$

$xC'' + C' = 4\ln x$

Let $z = C'$

$xz' + z = 4\ln x$

$z' + (\frac{1}{x})z = \frac{4}{x}\ln x$

The integrating factor is $e^{\int \frac{1}{x}dx}$

$= e^{\ln x}$

$= x$

$\therefore \quad xz' + z = 4\ln x$

$\frac{d}{dx}(xz) = 4\ln x$

$xz = \int 4\ln x\, dx$

$xz = 4x\ln x - \int 4\, dx$

$xz = 4x\ln x - 4x + B$

$z = 4\ln x - 4 + \frac{B}{x}$

$C' = 4\ln x - 4 + \frac{B}{x}$

$C(x) = 4x\ln x - 4x - 4x + B\ln x + D$

$= 4x\ln x - 8x + B\ln x + D$, B and D are arbitrary constants

$\therefore\ y(x) = \left(\frac{1}{x}\right)(4x\ln x - 8x + B\ln x + D)$

$= 4\ln x - 8 + \frac{B}{x}\ln x + \frac{D}{x}$ where B and D are arbitrary constants.

2(a)(i)

$$S_m = \sum_{n=2}^{m}\left(\frac{1}{\sqrt[3]{3n+2}} - \frac{1}{\sqrt[3]{3n+5}}\right)$$

$$= \frac{1}{\sqrt[3]{6+2}} - \frac{1}{\sqrt[3]{6+5}}$$

$$+ \frac{1}{\sqrt[3]{9+2}} - \frac{1}{\sqrt[3]{9+5}}$$

$$\vdots$$

$$+ \frac{1}{\sqrt[3]{3(m-1)+2}} - \frac{1}{\sqrt[3]{3(m-1)+5}}$$

$$+ \frac{1}{\sqrt[3]{3m+2}} - \frac{1}{\sqrt[3]{3m+5}}$$

$$= \frac{1}{\sqrt[3]{8}} - \frac{1}{\sqrt[3]{3m+5}}$$

$$= \frac{1}{2} - \frac{1}{\sqrt[3]{3m+5}} \quad \#$$

$$\therefore S_m = \frac{1}{2} - \frac{1}{\sqrt[3]{3m+5}}$$

(ii) Let $S_n = \frac{1}{2} - \frac{1}{\sqrt[3]{3n+5}}$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1}{2} - \frac{1}{\sqrt[3]{3n+5}}$$

$$= \frac{1}{2} - 0$$

$$= \frac{1}{2}$$

$\therefore$ The series is convergent and it has a limit of $\frac{1}{2}$.

(b) As $n \to \infty$, $\frac{1}{n} \to 0$ and $\sin\left(\frac{1}{3\ln n}\right) \to 0$. However, $\frac{1}{n}$ would approaches 0 much faster than $\sin\left(\frac{1}{3\ln n}\right)$. Hence, the series $\sum_{n=3}^{\infty} \frac{\sin\left(\frac{1}{3\ln n}\right)}{n}$ is convergent.

3. Initial ~~Volume~~ liter of mixture, $L_0 = 400\,L$

Initial Concentration of chlorine, $\frac{dy}{dt} = 0.05\ g/L$.

Initial amount of chlorine, $y_0 = 0.05 \times 400$

$= 20\,g$

~~Chage~~ Change in liter as a function of time $= 5 - 10 = -5\ L/s$

$$L = 400 - 5t$$

$$\frac{dy}{dt} = -\frac{10\,y(t)}{400-5t}$$

$$\int \frac{dy}{10y} = -\int \frac{dt}{400-5t}$$

$$\frac{1}{10}\ln y = \frac{1}{5}\ln|400-5t| + C$$

$$\because\ y(0) = 20\,g$$

$$\therefore\ \frac{1}{10}\ln 20 = \frac{1}{5}\ln|400| + C$$

$$C = \frac{1}{10}\ln 20 - \frac{1}{5}\ln 400$$

$$\therefore\ \frac{1}{10}\ln y = \frac{1}{5}\ln|400-5t| + \frac{1}{10}\ln(20) - \frac{1}{5}\ln 400$$

$$\ln y = 2\ln|400-5t| + \ln(20) - 2\ln(400)$$

$$\ln y = \ln \frac{(400-5t)^2(20)}{400^2}$$

$$y = \frac{(400-5t)^2(20)}{400^2}$$

$$y(t_0) = 0$$

$$0 = \frac{(400-5t_0)^2(20)}{400^2}$$

$$400 - 5t_0 = 0$$

$$t_0 = 80\,s\ \#$$

$t = \frac{1}{2}t_0$

$= \frac{1}{2}(80)$

$= 40\,s$

$$y = \frac{[400-5(40)]^2(20)}{400^2}$$

$= 5\,g$ #

$\therefore$ $t_0 = 80\,s$, $y = 5\,g$ at time $t = \frac{1}{2}t_0$.

4 Let $M = \sin(xy) + xy\cos(xy) + y^2 + e^{3x}$

$N = x^2\cos(xy) + 2xy$

$M_y = \frac{\partial M}{\partial y} = x\cos(xy) + x\cos(xy) - x^2y\sin(xy) + 2y$

$= 2x\cos(xy) - x^2y\sin(xy) + 2y$

$N_x = \frac{\partial N}{\partial x} = 2x\cos(xy) - x^2y\sin(xy) + 2y$

Since $M_y = N_x$, it is verified that the differential equation is exact.

Let $\frac{\partial u}{\partial x} = M = \sin(xy) + xy\cos(xy) + y^2 + e^{3x}$

$\frac{\partial u}{\partial y} = N = x^2\cos(xy) + 2xy$

$u(x,y) = \int x^2\cos(xy) + 2xy \, dy$

$= x\sin(xy) + xy^2 + f(x)$, $f(x)$ is an arbitrary function of x

$\frac{\partial u}{\partial x} = \sin(xy) + xy\cos(xy) + y^2 + f'(x)$

By comparison, $f'(x) = e^{3x}$

$f(x) = \frac{1}{3}e^{3x}$

$\therefore u(x,y) = x\sin(xy) + xy^2 + \frac{1}{3}e^{3x}$

$\therefore x\sin(xy) + xy^2 + \frac{1}{3}e^{3x} = C$ where C is an arbitrary constant.

5(a) $T_3(x) = f(0) + \frac{f'(0)}{1!}(x-0) + \frac{f^{(2)}(0)}{2!}(x-0)^2 + \frac{f^{(3)}(0)}{3!}(x-0)^3$

$= 2 + \frac{1}{4}x - \frac{1}{64}x^2 + \frac{1}{512}x^3$

$f(0) = \sqrt{4+0} = 2$

$f'(x) = \frac{1}{2}(4+x)^{-\frac{1}{2}}$

$f'(0) = \frac{1}{4}$

$f^{(2)}(x) = -\frac{1}{4}(4+x)^{-\frac{3}{2}}$

$f^{(2)}(0) = -\frac{1}{32}$

$f^{(3)}(x) = \frac{3}{8}(4+x)^{-\frac{5}{2}}$

$f^{(3)}(0) = \frac{3}{256}$

$\therefore T_3(x) = 2 + \frac{1}{4}x - \frac{1}{64}x^2 + \frac{1}{512}x^3$

(b) $4 + x = 3.6$

$x = -0.4$

$T_3(-0.4) = 2 + \frac{1}{4}(-0.4) - \frac{1}{64}(-0.4)^2 + \frac{1}{512}(-0.4)^3$

$= 1.897375$

$\therefore \sqrt{3.6} \simeq T_3(-0.4) = 1.897375$

(c) $T_2(x) = 2 + \frac{1}{4}x - \frac{1}{64}x^2$

$R_2(x) = \frac{f^{(3)}(c)}{3!}(x-0)^3$

$= \frac{\frac{3}{8}(4+c)^{-\frac{5}{2}}}{6}x^3$

$= \frac{(4+c)^{-\frac{5}{2}}}{16}x^3$

$\because |x| < 0.2$

$-0.2 < x < 0.2$

$a < c < x$

$0 < c < 0.2$

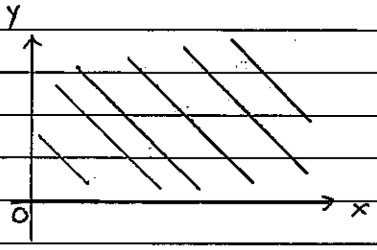
$|R_2(0.2)| = \frac{(4+c)^{-\frac{5}{2}}}{16}(0.2)^3$

$= \frac{(4+c)^{-\frac{5}{2}}}{2000} < \frac{(4+0.2)^{-\frac{5}{2}}}{2000} = 1.383\times10^{-5} < 2\times10^{-4}$

$\therefore$ The error, $|R_2(0.2)| < 2\times10^{-4}$.

6(a) $xy > 0$ for f to be defined.

$\therefore \quad D = \{(x,y) : x>0 \cap y>0\}$.



exclusive of y and x axis.

(b) $D_v f(1,4) = \nabla f(1,4) \cdot \hat{v}$

$\nabla f(x,y) = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle$

$\frac{\partial f}{\partial x} = -\frac{2}{y}x^{-3} + \frac{1}{2}(xy)^{-\frac{1}{2}}(y)$

$\frac{\partial f}{\partial y} = -\frac{1}{x^2}y^{-2} + \frac{1}{2}(xy)^{-\frac{1}{2}}(x)$

$\nabla f(1,4) = \langle \frac{1}{2}, \frac{3}{16} \rangle$

$\hat{v} = \frac{1}{\sqrt{(1)^2+(-1)^2}}(1,-1)$

$= \frac{1}{\sqrt{2}}(1,-1)$

$D_v f(1,4) = \langle \frac{1}{2}, \frac{3}{16} \rangle \cdot \frac{1}{\sqrt{2}}(1,-1)$

$= \frac{5\sqrt{2}}{32}$

$\therefore$ Rate of change in f, $D_v f(1,4) = \frac{5\sqrt{2}}{32}$.

(c) $\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$

$\frac{\partial x}{\partial t} = 2e^{s+2t}$

$\frac{\partial y}{\partial t} = 2t$

when $s=-2$, $t=1$

$\frac{\partial x}{\partial t} = 2 \qquad x = 1$

$\frac{\partial y}{\partial t} = 2 \qquad y = 5$

$\frac{\partial f}{\partial x} = -\frac{2}{5}(1)^{-3} + \frac{1}{2}(5)^{-\frac{1}{2}}(5)$

$= -\frac{2}{5} + \frac{5}{2}(5)^{-\frac{1}{2}}$

$\frac{\partial f}{\partial y} = -\frac{1}{(1)^2}(5)^{-2} + \frac{1}{2}(5)^{-\frac{1}{2}}(1)$

$= -\frac{1}{25} + \frac{1}{2}(5)^{-\frac{1}{2}}$

$\frac{\partial f}{\partial t} = -\frac{4}{5} + 5(5)^{-\frac{1}{2}} - \frac{2}{25} + (5)^{-\frac{1}{2}}$

$= -\frac{22}{25} + 6(5)^{-\frac{1}{2}} \qquad \therefore \frac{\partial f}{\partial t} = -\frac{22}{25} + 6(5)^{-\frac{1}{2}}$

7(a)

$$\lim_{\substack{(x,y)\to(0,0)\\ y\geq 0}} x+\sqrt{y} = 0$$

$$\lim_{\substack{(x,y)\to(0,0)\\ y<0}} \frac{x^6-4x^2y^2}{x^4y+3y^3} = \lim_{\substack{(x,y)\to(0,0)\\ y<0\\ y=x}} \frac{x^6-4x^4}{x^5+3x^3} = \lim_{\substack{(x,y)\to(0,0)\\ y<0\\ y=x}} \frac{x^3-4x}{x^2+3} = \frac{0-0}{0+3} = 0$$

$\therefore$ The limit $\lim_{(x,y)\to(0,0)} f(x,y)$ exist and the limit is 0.

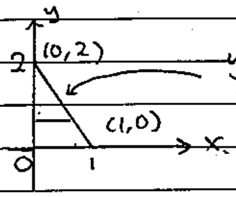
(b) when $y \geq 0$, $f(x,y) = x+\sqrt{y}$

$f_y(x,y) = \frac{1}{2}(y)^{-\frac{1}{2}}$

$f_y(0,0) = \frac{1}{2}(0)^{-\frac{1}{2}}$, Does not exist.

$\therefore$ $f_y(0,0)$ does not exist.

(c)



$y = -2x+2$

$x = 1-\frac{1}{2}y$, $y \geq 0$.

$$\iint_R f(x,y)\,dx\,dy$$

$$= \int_0^2 \int_0^{1-\frac{1}{2}y} x+\sqrt{y}\,dx\,dy$$

$$= \int_0^2 \left[\frac{1}{2}x^2 + x\sqrt{y}\right]_0^{1-\frac{1}{2}y} dy$$

$$= \int_0^2 \frac{1}{2}(1-\tfrac{1}{2}y)^2 + (1-\tfrac{1}{2}y)\sqrt{y}\,dy$$

$$= \int_0^2 \frac{1}{2}(1-y+\tfrac{1}{4}y^2) + \sqrt{y} - \tfrac{1}{2}y^{3/2}\,dy$$

$$= \left[\tfrac{1}{2}y - \tfrac{1}{4}y^2 + \tfrac{1}{24}y^3 + \tfrac{2}{3}y^{3/2} - \tfrac{1}{5}y^{\frac{5}{2}}\right]_0^2$$

$$= 1.0876$$

8. $f(x,y) = (2x-x^2)(y^2-4y)$, $S = \{(x,y) : x^2+y^2 > 4\}$

$= 2xy^2 - 8xy - x^2y^2 + 4x^2y$

$\nabla f(x,y) = \langle 2y^2 - 8y - 2xy^2 + 8xy,\ 4xy - 8x - 2x^2y + 4x^2 \rangle$

Let $\nabla f(x,y) = 0$

$2y^2-8y-2xy^2+8xy=0$	$4xy-8x-2x^2y+4x^2=0$
$2y-8-2xy+8x=0.$	$4y-8-2xy+4x=0$
$2y-8 = x(2y-8)$	$4y-8 = x(2y-4)$
$x=1$	$x=2$
when $x=1$, $4y-8-2(1)y+4(1)=0$	when $x=2$, $2y-8-2(2)y+8(2)=0$
$2y=4$	$y=4$
$y=2$	

$\therefore$ (1, 2) and (2, 4) are stationary points.

$f_x = \frac{\partial f}{\partial x} = 2y^2 - 8y - 2xy^2 + 8xy$ $\quad$ $f_y = \frac{\partial f}{\partial y} = 4xy - 8x - 2x^2y + 4x^2$

$f_{xx} = -2y^2 + 8y$ $\quad$ $f_{yy} = 4x - 2x^2$

$f_{xy} = 4y - 8 - 4xy + 8x$ $\quad$ $f_{yx} = f_{xy}$

$D = f_{xx} f_{yy} - (f_{xy})^2$

At point (1, 2)

$f_{xx} = -2(2)^2 + 8(2) = 8 > 0$

$f_{yy} =$ ~~4(2) - 8 4(1)(2) + 8(1) = 0~~ $4(1) - 2(1)^2 = 2$

$f_{xy} = 4(2) - 8 - 4(1)(2) + 8(1) = 0$

$D =$ ~~8(0) = 0~~ $8(2) - 0^2$

$= 16 > 0$

(1, 2) is a local minimum point. because D > 0 and $f_{xx} > 0$.

At point (2, 4)

$f_{xx} = -2(4)^2 + 8(4) = 0$

$f_{yy} =$ ~~4(4) - 2(4)^2 =~~ $4(2) - 2(2)^2 = 0$

$f_{xy} = 4(4) - 8 - 4(2)(4) + 8(2) = -8$

$D = 0(0) - (-8)^2$

$= -64 < 0$

(2, 4) is a saddle point because D < 0.

$\therefore$ (1, 2) is a local minimum point and (2, 4) is a saddle point.

DISCLAIMER: *The solutions are done by students who scored A or above in this subject. The MAE Club and Campus supplies are not liable or responsible for any errors in the contents of these solutions. Students are advised to take the solutions as a guide rather than absolute answers to exam paper.*

NANYANG TECHNOLOGICAL UNIVERSITY
SEMESTER 1 EXAMINATION 2015–2016
MH1811 – MATHEMATICS 2

December 2015 TIME ALLOWED: 2 HOURS

Matriculation Number:

Seat Number:

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **EIGHT (8)** questions and comprises **Twelve (12)** pages, including an Appendix.
2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
3. This **IS NOT** an **OPEN BOOK** exam. However, a list of formulae is provided in the Appendix.
4. Candidates may use calculators. However, they should write down systematically the steps in the workings.
5. All your solutions should be written in this booklet within the space provided after each question. If you use an additional answer book, attach it to this booklet and hand them in at the end of the examination.

For examiners only

Question	Marks
1 (15)	
2 (15)	
3 (15)	
4 (10)	

Question	Marks
5 (15)	
6 (10)	
7 (10)	
8 (10)	

TOTAL (100)

3

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QUESTION 1. (15 Marks)

(a) Solve the homogeneous differential equation

$$y'' + y = 0.$$

(b) Use the method of Variation of Parameters to find a general solution on $(-\frac{\pi}{2}, \frac{\pi}{2})$ to the following differential equation

$$y'' + y = \tan x.$$

End of Question 1.

QUESTION 2. (15 Marks)

(a) Find a formula for the sum $\sum_{n=3}^{m}\left(\frac{1}{\sqrt{3n}}-\frac{1}{\sqrt{3(n+1)}}\right)$ in terms of m. (You may leave your answer in surd form.)

(b) Use part(a) to determine whether the series $\sum_{n=3}^{\infty}\left(\frac{1}{\sqrt{3n}}-\frac{1}{\sqrt{3(n+1)}}\right)$ converges or diverges. If it converges, find its limit. Justify your answers.

End of Question 2.

QUESTION 3. (15 Marks)

Let $f(x,y)=\frac{x-1}{x^2+y^2-1}$.

(a) Find the domain for the function $f(x,y)$, and sketch the domain of f on an xy-plane.

(b) Determine whether $\lim_{(x,y)\to(1,0)} f(x,y)$ exists. If it exists, determine its limit. Justify your answer.

Question 3 continues on page 5.

(c) Determine the rate of change in f at $(0,0)$ along the direction $\mathbf{v} = \mathbf{i} + \mathbf{j}$.

End of Question 3.

QUESTION 4. (10 Marks)

Consider the differential equation

$$y'' - \left(\frac{1}{x}\right) y' + \left(\frac{1}{x^2}\right) y = 0. - - - (*)$$

(a) Verify that $y_1(x) = x$ is a solution of the differential equation(*).

(b) Use part(a) to determine a second linearly independent solution of the differential equation(*) for $x > 0$.

End of Question 4.

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QUESTION 5. (15 Marks)

Let $f(x) = \dfrac{1}{\sqrt{1+x}}$.

(a) Determine the Taylor series, up to x^4, of $f(x)$ at $a = 0$.

(b) The value $f(x)$ is estimated by by $T_2(x)$ for $|x| < 0.01$, determine the error using the Taylor's remainder.

End of Question 5.

QUESTION 6. (10 Marks)

Consider a closed rectangular box with a square base of side 3 cm, and height 5 cm. If the side is measured with an error at most 0.02 cm and the height is measured with an error at most 0.01 cm, use differentials to estimate the maximum possible error in computing the volume of the box.

End of Question 6.

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QUESTION 7. (10 Marks)

(a) Let R be the region bounded by $y = x^2$, the y-axis for $0 \leq x \leq 2$. Sketch the region R.

(b) Evaluate $\iint\limits_R \frac{xy}{\sqrt{1+x^2+y^2}} dA$ over the region R given in Part(a).

End of Question 7.

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QUESTION 8. (10 Marks)

Find the global (absolute) maximum and minimum of $f(x,y) = xy - x - 3y$ on the the triangular region R with vertices $(0,0)$, $(0,4)$, and $(5,0)$.

End of Question 8.

END OF PAPER

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Appendix

Derivatives

$\frac{d}{dx}(x^n) = nx^{n-1}$

$\frac{d}{dx}(\sin x) = \cos x$ $\quad$ $\frac{d}{dx}(\cos x) = -\sin x$

$\frac{d}{dx}(\tan x) = \sec^2 x$ $\quad$ $\frac{d}{dx}(\cot x) = -\csc^2 x$

$\frac{d}{dx}(\sec x) = \sec x \tan x$ $\quad$ $\frac{d}{dx}(\csc x) = -\csc x \cot x$

$\frac{d}{dx}(e^x) = e^x$ $\quad$ $\frac{d}{dx}(\ln x) = \frac{1}{x}$

$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$ $\quad$ $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

$\frac{d}{dx}(\sinh x) = \cosh x$ $\quad$ $\frac{d}{dx}(\cosh x) = \sinh x$

$\frac{d}{dx}(\tanh x) = \text{sech}^2 x$ $\quad$ $\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{x^2+1}}$

Antiderivatives

$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$ $\quad$ $\int \frac{1}{x}\,dx = \ln|x| + C$

$\int \sin x\,dx = -\cos x + C$ $\quad$ $\int \cos x\,dx = \sin x + C$

$\int \sec^2 x\,dx = \tan x + C$ $\quad$ $\int \csc^2 x\,dx = -\cot x + C$

$\int \tan x \sec x\,dx = \sec x + C$ $\quad$ $\int \cot x \csc x\,dx = -\csc x + C$

$\int \tan x\,dx = \ln|\sec x| + C$ $\quad$ $\int e^x\,dx = e^x + C$

$\int \sec x\,dx = \ln|\sec x + \tan x| + C$ $\quad$ $\int \csc x\,dx = -\ln|\csc x + \cot x| + C$

$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}(x) + C$ $\quad$ $\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$

Trigonometry

$\sin^2\theta + \cos^2\theta = 1$ $\quad$ $1 + \tan^2\theta = \sec^2\theta$

$\sin(2\theta) = 2\sin\theta\cos\theta$ $\quad$ $\cos(2\theta) = \cos^2\theta - \sin^2\theta$

$\sin(A+B) = \sin A\cos B + \cos A\sin B$ $\quad$ $\sin(A-B) = \sin A\cos B - \cos A\sin B$

$\cos(A+B) = \cos A\cos B - \sin A\sin B$ $\quad$ $\cos(A-B) = \cos A\cos B + \sin A\sin B$

$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$

Variation of Parameters.

$$y_p = C_1(x)y_1(x) + C_2(x)y_2(x) \text{ where } \begin{cases} C_1'y_1 + C_2'y_2 &= 0 \\ C_1'y_1' + C_2'y_2' &= F/a \end{cases}.$$

Taylor Series of $f(x)$ about a.

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

Taylor Remainder

$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$, for some real number c between x and a.

Maclaurin Series of some functions.

$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots \qquad (|x| < 1)$

$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \qquad (|x| < \infty)$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots \qquad (|x| < \infty)$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots \qquad (|x| < \infty)$

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{n+1} x^n}{n} + \cdots \qquad (-1 < x \leq 1)$

Linearization of $f(x,y)$ about (a,b)

$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

Q1

(a) $y'' + y = 0$

$\lambda^2 + 1 = 0 \Rightarrow \lambda^2 = -1$

$\therefore \lambda = \pm i$

$y_1 = \cos x \quad y_2 = \sin x$

$y = C_1 \cos x + C_2 \sin x$

(b) for $y'' + y = 0$

$y_h = C_1 \cos x + C_2 \sin x$

Replace constant C_1, C_2 by function $C_1(x), C_2(x)$

$y_p = C_1(x) \cos x + C_2(x) \sin x$

where $C_1(x), C_2(x)$ satisfy:

$$\begin{cases} \cos x\, C_1' + \sin x\, C_2' = 0 \\ -\sin x\, C_1' + \cos x\, C_2' = \tan x \end{cases} \qquad C_2' = C_1' \cdot \left(-\frac{\cos x}{\sin x}\right)$$

$$C_1' \left(\frac{-\sin^2 x}{\sin x} + \frac{-\cos^2 x}{\sin x}\right) = \tan x$$

$$C_1' = -\tan x \cdot \sin x = -\sin^2 x \cos x$$

$$\therefore C_2' = \sin x \cos^2 x$$

$$C_1 = \int -\sin^2 x \cos x\, dx = \int -\sin^2 x\, d(\sin x) = -\frac{1}{3}\sin^3 x + C_3$$

$$C_2 = \int \sin x \cos^2 x\, dx = \int -\cos^2 x\, d(\cos x) = -\frac{1}{3}\cos^3 x + C_4$$

$$\therefore y_p = -\frac{1}{3}\sin^3 x \cos x - \frac{1}{3}\cos^3 x \sin x + C_5$$

$$\therefore y = y_h + y_p = C_1 \cos x + C_2 \sin x - \frac{1}{3}\sin^3 x \cos x - \frac{1}{3}\cos^3 x \sin x + C_5$$

Q2

(a) $\sum_{n=3}^{m} \left(\frac{1}{\sqrt{3n}} - \frac{1}{\sqrt{3(n+1)}}\right)$

$= \frac{1}{\sqrt{3\times 3}} - \frac{1}{\sqrt{3\times 4}}$

$+ \frac{1}{\sqrt{3\times 4}} - \frac{1}{\sqrt{3\times 5}}$

$\vdots$

$+ \frac{1}{\sqrt{3m}} - \frac{1}{\sqrt{3(m+1)}}$

$= \frac{1}{3} - \frac{1}{\sqrt{3(m+1)}}$

(b) when $n \to \infty$

$\sum_{n=3}^{\infty}\left(\frac{1}{\sqrt{3n}} - \frac{1}{\sqrt{3(n+1)}}\right) \to \frac{1}{3}$ as $\frac{1}{\sqrt{3(n+1)}} \to 0$

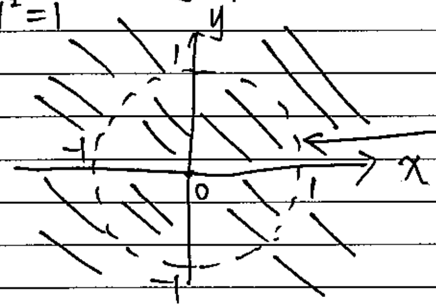
∴ the series converges to $\frac{1}{3}$

Q3

(a) $f(x,y) = \frac{x-1}{x^2+y^2-1}$

$x^2+y^2 \neq 1$

the domain is every point on the xy plance except those points on $x^2+y^2=1$



exclusive of $x^2+y^2=1$

(b) $\lim_{\substack{(x=y+1)\\(y=0)}} f(x,y) = \frac{y}{y^2+2y+1+y^2-1} = \frac{y}{2y^2+2y} = \frac{1}{2y+2} = \frac{1}{2}$

$\lim_{\substack{(y=x-1)\\(x=1)}} f(x,y) = \frac{x-1}{x^2+x^2-2x+1-1} = \frac{x-1}{2x^2-2x} = \frac{x-1}{2x(x-1)} = \frac{1}{2x} = \frac{1}{2}$

∵ $\lim_{\substack{(x=y+1)\\(y=0)}} f(x,y) = \lim_{\substack{(y=x-1)\\(x=1)}} f(x,y)$

∴ $\lim_{(x,y)\to(1,0)} (x,y)$ exists

(c) $D_u f(x,y) = \nabla f \cdot u = f_x(x,y)\,\hat{i} + f_y(x,y)\,\hat{j}$

$= \frac{2x(x-1)-(x^2+y^2-1)}{(x^2+y^2-1)^2}\,\hat{i} + \frac{2y(x-1)}{(x^2+y^2-1)^2}\,\hat{j}$

when $x=0$, $y=0$

$D_u f(x,y) = 1\,\hat{i} - 1\,\hat{j}$

$= \hat{i} - \hat{j}$

Q4

(a) When $y_1 = x$, $y_1' = 1$ $y_1'' = 0$

sub into (*): $0 - (\frac{1}{x})1 + (\frac{1}{x^2})(x) = -\frac{1}{x} + \frac{1}{x} = 0$

$\therefore$ $y_1(x) = x$ is a solution of (*)

(b) let $y_2 = C(x)x$

$y_2' = C(x) + C'(x)x$

$y_2'' = C'(x) + C'(x) + C''(x) = C''(x) + 2C'(x)$

sub into (*): $C''(x) + 2C'(x) - \frac{1}{x}(C(x) + C'(x)x) + (\frac{1}{x^2})(C(x)x) = 0$

$C''(X) + 2C'(x) - \frac{1}{x}C(x) + C'(x) + \frac{1}{x}C(x) = 0$

$C''(x) + 3C'(x) = 0 \Rightarrow \lambda^2 + 3\lambda = 0$

$\because$ $3^2 - 4(1) = 5 > 0$

$\therefore$ $\lambda_1 = -3, \lambda_2 = 0$

$y_3 = e^{-3x}, y_4 = 1$

$y_2 = C_1e^{-3x} + C_2$

another linearly independent solution is $C_1e^{-3x} + C_2$ $(x > 0)$

Q5

(a)

$f(x) = (1+x)^{-\frac{1}{2}}$	$f(0) = 1$
$f'(x) = -\frac{1}{2}(1+x)^{-\frac{3}{2}}$	$f'(0) = -\frac{1}{2}$
$f''(x) = \frac{3}{4}(1+x)^{-\frac{5}{2}}$	$f''(0) = \frac{3}{4}$
$f^{(3)}(x) = -\frac{15}{8}(1+x)^{-\frac{7}{2}}$	$f^{(3)}(0) = -\frac{15}{8}$
$f^{(4)}(x) = \frac{105}{16}(1+x)^{-\frac{9}{2}}$	$f^{(4)}(0) = \frac{105}{16}$

$T(x) = f(0) + \frac{f'(0)}{1}(x) + \frac{f''(0)}{2}(x^2) + \frac{f^{(3)}(0)}{6}(x^3) + \frac{f^{(4)}(0)}{24}(x^4) + \ldots$

$= 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \frac{35}{128}x^4 + \ldots$

(b) $f(x) = T_2(x) + R_2(x)$

$R_2(x) = T_3(x) = -\frac{15}{8}(1+0.01)^{-\frac{7}{2}}$

$= -1.8108$

Q6 $V = a^2 h$

$da = 0.02, \quad dh = 0.01$

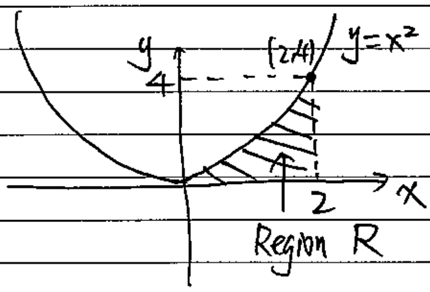
$dV = \frac{\partial V}{\partial a} da + \frac{\partial V}{\partial h} dh$

$dV = 2ah\,da + a^2 dh$

$= 2(3)(5)(0.02) + (0.02)^2 0.01$

$= 0.600004$

Q7 (a)



(There is a change during the test where the boundary is changed to area between x-axis, x=2 and $y = x^2$

~~(b) $\iint_R \frac{xy}{\sqrt{1+x^2+y^2}} dA$~~

~~$\int_0^2 \int_0^{x^2} \frac{xy}{\sqrt{1+x^2+y^2}} dy\,dx$~~ $= \int_0^4 \int_{0}^{2} \frac{xy}{\sqrt{1+x^2+y^2}} dx\,dy$

~~$= \int_0^2 \left[x(1+x^2+y^2)^{\frac{1}{2}} \right]_0^{x^2} dx$~~ ~~$= \int_0^4 \left[y(1+x^2+y^2)^{\frac{1}{2}} \right]_0^2 dy$~~

~~$= \int_0^2 x(1+x^2+x^4)^{\frac{1}{2}} - x(1+x^2)^{\frac{1}{2}} dx$~~ ~~$= \int_0^4 y(5+y^2)^{\frac{1}{2}} - y(1+y^2)^{\frac{1}{2}} dy$~~

~~(b) $\iint_R \frac{xy}{\sqrt{1+x^2+y^2}} dA$~~

~~$= \int_0^y \int_0^x \frac{xy}{\sqrt{1+x^2+y^2}} dy\,dx$~~

~~$= \int_0^4 \left[y\sqrt{1+x^2+y^2} \right]_0^{\sqrt{y}} dy$~~

~~$= \int_0^4 y\sqrt{1+x^2+y^2} - y\sqrt{1+y^2}\, dy$~~

~~$= \frac{1}{3}(1+x^2+y^2)^{\frac{3}{2}} - \frac{1}{3}(1+y^2)^{\frac{1}{2}} \Big|_0^4$~~

Q8.

$f(x,y) = xy - x - 3y$

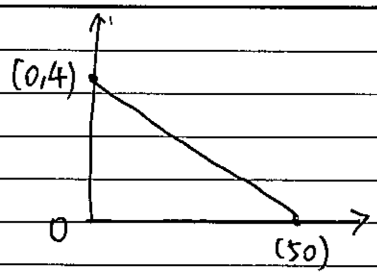
$f_x(x,y) = y - 1$

$f_y(x,y) = x - 3$

when $f_x(x,y) = f_y(x,y) = 0$

$y = 1,\ x = 3$

$\therefore f(x,y) = 3 - 3 - 1 \times 3 = -3$, which is a local extreme value.



At (0,4), $f(x,y) = -12$

At (0,0), $f(x,y) = 0$

At (5,0), $f(x,y) = -5$

$\therefore$ global maximum is 0 while global minimum is -12.

NANYANG TECHNOLOGICAL UNIVERSITY
SEMESTER 2 EXAMINATION 2015–2016
MH1811 – MATHEMATICS 2

APRIL 2016 TIME ALLOWED: 2 HOURS

Matriculation Number:

Seat Number:

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **SIX (6)** questions and comprises **THIRTEEN (13)** pages.
2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
3. This is a **RESTRICTED OPEN BOOK** exam. Each candidate is allowed to bring a hand-written A4 help sheet.
4. Candidates may use calculators. However, they should write down systematically the steps in the workings.
5. All your solutions should be written in this booklet within the space provided after each question. However, if you write your solutions on other pages, please indicate them clearly.

For examiners only

Question	Marks
1 (15)	
2 (20)	
3 (15)	

Question	Marks
4 (15)	
5 (15)	
6 (20)	

TOTAL (100)

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QUESTION 1. (15 Marks)

(a) Solve the initial value problem.

$$yy' = xe^{2x-y^2}, y(0.5) = 1.$$

Question 1 continues on page 3.

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(b) Solve the non-homogeneous differential equation.

$$y'' - 3y' = e^{3x}$$

End of Question 1.

QUESTION 2. (20 Marks)

(a) A tank with capacity of 1000 L contains 200 L of brine solution with a concentration of 0.2 kg/L. A brine solution with a salt concentration of 0.4 kg/L is added at a rate of 15 L/min. The solution is kept mixed and is drained from the tank at a rate of 5 L/min. Let $y(t)$ be the amount of salt (in kilogrammes) in the tank after t minutes. Find the amount of salt in the solution after 20 minutes. Leave your answer up to three decimal places.

Question 2 continues on page 5.

(b) Consider the function $f(x, y) = \ln(x^2 - 3x - y)$.

(i) Determine the domain of $f(x, y)$ and indicate it on Figure 1.

Domain of f: $\{(x, y) \in \mathbb{R}^2 \mid$ ______________________$\}$

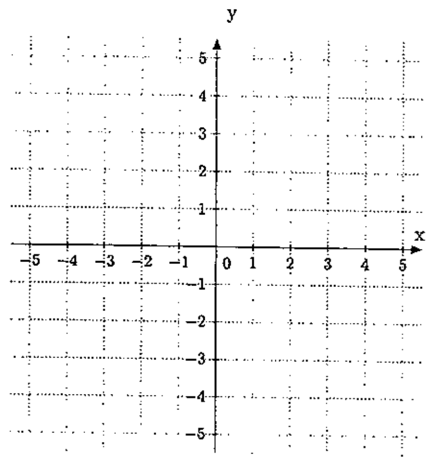

Figure 1.

(ii) Find the value c such that $f(4, c) = 0$. With this value of c, evaluate $\nabla f(4, c)$ and indicate its direction at the point $(4, c)$ on Figure 1.

End of Question 2.

QUESTION 3. (15 Marks)

Consider the function $f(x, y) = 9x - x^3 + y^2 - 7y$.

(a) Write down the equation of the tangent plane to the graph $z = f(x, y)$ at the point $P(2, 4)$. Leave your equation in the form $z = ax + by + c$, where a, b and c are some constants.

Equation of the tangent plane is $z =$ ______________________.

(b) Find and classify all stationary points of the function $f(x, y) = 9x - x^3 + y^2 - 7y$.

Question 3 continues on page 7.

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(c) The length x of a side of a triangle is increasing at a rate of 6 cm/s, the length y of another side is decreasing at a rate of 4 cm/s, and the contained angle θ is decreasing at a rate of 0.05 radian/s. How fast is the area of the triangle changing when $x = 80$ cm, $y = 100$ cm and $\theta = \pi/6$? Is it increasing or decreasing? Leave your answer up to 2 decimal places.

End of Question 3.

QUESTION 4. (15 Marks)

(a) Let $T(x, y, z) = 6x - 3y + 2z$ be the temperature (in degree Celsius) of a point (x, y, z) on the surface S defined by $(x-1)^2 + 3y^2 + z^2 = 43$. Assume the highest temperature exists, use Lagrange Multiplier Method to determine the point(s) on S with the highest temperature.

Question 4 continues on page 9.

(b) Consider a quadrilateral Q on the first quadrant of the xy-plane, with vertices $(0,0)$, $(0,4)$, $(5,4)$ and $(2,1)$. (A quadrilateral is a 4-sided flat figure.)

Indicate the region D bounded by the quadrilateral Q on Figure 2 and determine the volume of the solid under the surface $z = x^2$ and above the region D.

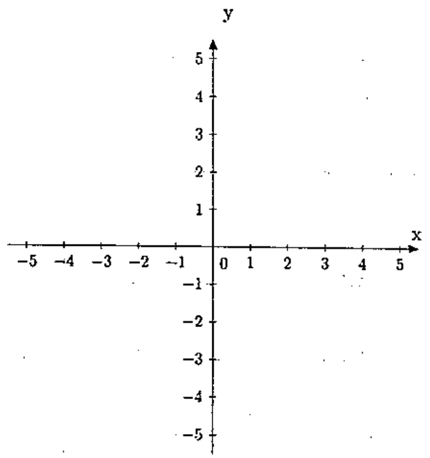

Figure 2.

End of Question 4.

QUESTION 5. (15 Marks)

(a) Suppose $0 < a < 1$. Use Squeeze Theorem to determine the limit $\lim_{n\to\infty} \sqrt[n]{1+a^n}$.

(b) Use your result in part (a) to determine $\lim_{n\to\infty} \sqrt[n]{3^n + 7^n}$.

Question 5 continues on page 11.

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(c) Consider the power series $\sum_{n=0}^{\infty}(3^n + 7^n)\, x^n$. Determine its interval of convergence and its sum. Justify your answer.

Interval of Convergence: ____________________

Sum: ____________________

Explanation:

End of Question 5.

QUESTION 6. (20 Marks)

(a) Determine whether the series $\sum_{n=0}^{\infty} \frac{(-1)^n (n^2)}{5+n^3}$ is absolutely convergent, conditionally convergent or divergent. Justify your answer.

Question 6 continues on page 13.

(b) Find the Taylor series (up to and including x^3) of $f(x) = \dfrac{1}{4 + x - x^2}$ about $a = 0$, and use it to to approximate $\dfrac{1}{4.09}$. Leave your answer up to 4 decimal places.

End of Question 6.

END OF PAPER

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1. a) $yy' = xe^{2x-y^2}$, $y(0.5) = 1$

$yy' = xe^{2x}/e^{y^2}$ (*)

$\int e^{y^2} y\, dy = \int xe^{2x} dx$ | let $u = x$, $v' = e^{2x}$

$\frac{1}{2}e^{y^2} = \int xe^{2x} dx$ .... (*) $du = dx$, $v = \frac{1}{2}e^{2x}$

$\frac{1}{2}e^{y^2} = \frac{1}{2}e^{2x}x - \int \frac{1}{2}xe^{2x} dx$

$= \frac{1}{2}(xe^{2x} - \frac{1}{2}e^{2x} + c)$

$e^{y^2} = xe^{2x} - \frac{1}{2}e^{2x} + c$

now, $y(0.5) = 1$

$e = 0.5e^{2(0.5)} - \frac{1}{2}e^{2(0.5)} + c$

$C = e$

$\therefore e^{y^2} = xe^{2x} - \frac{1}{2}e^{2x} + e$ ✱

b) $y'' - 3y' = e^{3x}$ —— (*)

let associated homogeneous differential equation be $\lambda'' - 3\lambda' = 0$

Solve, $\lambda_1 = 3$, $\lambda_2 = 0$

$\therefore y_h = C_1e^{3x} + C_2$

Using Undetermined Coeff. Method,

let $y_p = Axe^{3x}$

$y_p' = A(e^{3x} + 3xe^{3x})$

$y_p'' = A(3e^{3x} + 3(3)e^{3x}x + 3e^{3x})$

Subs all into (*) $A(6e^{3x} + 9xe^{3x}) - 3A(e^{3x} + 3xe^{3x}) = e^{3x}$

compare coeff. of e^{3x}, $6A - 3A = 1$

$A = \frac{1}{3}$

$\therefore y_g = y_p + y_h = \frac{1}{3}xe^{3x} + C_1e^{3x} + C_2$, where C_1 & C_2 are arbitrary const

2. a) Initial mixture, $L_0 = 200L$

Initial concentration of salt, $\frac{dy}{dL} = 0.2$ kg/L

Initial amount of salt $\Rightarrow y_0 = 0.2 \times 200 = 40$kg

Change in liter : $15 - 5 = 10L/min$ (in flow)

let amount of salt as a function of time, $y(t)$, & $L = 200 + 10t$

$\frac{dy}{dt}$ = ~~inflow - outflow~~

$= 0.4 \times 15 - \frac{y(t)}{200+10t}(5)$ $\Rightarrow y' + \frac{5y}{200+10t} = 6$

$= 6 - \frac{5y(t)}{200+10t}$

let $I(t) = e^{\int \frac{5}{200+10t} dt} = e^{\frac{5}{10}\ln(200+10t)} = (200+10t)^{0.5}$

DISCLAIMER: *The solutions are done by students who scored A or above in this subject. MAE Club and Campus supplies are not liable or responsible for any errors in the contents of these solutions. Students are advised to take the solutions as a guide rather than absolute answers to exam paper.*

P(1)

Continues 2) a.

$$y(t) \times (200+10t)^{0.5} = \int 6(200+10t)^{0.5}\,dt$$
$$= 6\int(200+10t)^{0.5}\,dt$$
$$= \frac{6}{10}\left(\frac{2}{3}\right)(200+10t)^{\frac{3}{2}} + C$$
$$= \frac{2}{5}(200+10t)^{\frac{3}{2}} + C$$

$\because t=0,\ y_0 = 40$

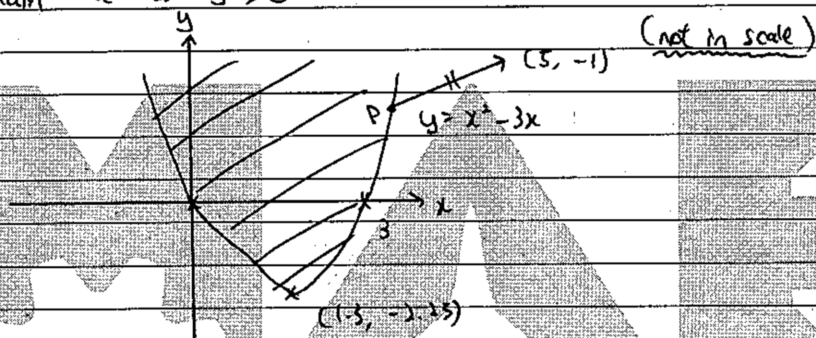
$$\therefore 40(200)^{0.5} = \frac{2}{5}(200)^{\frac{3}{2}} + C$$
$$C = -565.685$$

$$\therefore y(t) = \frac{1}{(200+10t)^{0.5}}\left[\frac{2}{5}(200+10t)^{\frac{3}{2}} - 565.685\right]$$

Now, $t = 20$, $y(20) = 131.716$ kg ✗

b) $f(x,y) = \ln(x^2 - 3x - y)$

Domain: $x^2 - 3x - y > 0$



ii) $f(4,c) = 0 \Rightarrow 0 = \ln(4^2 - 3(4) - c)$

$c = 3$ ✗

$$\nabla f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

$$\frac{\partial f}{\partial x} = \frac{1}{x^2 - 3x - y}(2x-3)$$

$$\frac{\partial f}{\partial y} = \frac{1}{x^2 - 3x - y}(-1)$$

At $P(4,3)$, $\nabla f(4,3) = (5, -1)$ ✗

3 a) $f(x,y) = 9x - x^3 + y^2 - 7y$

eqn. of tgt. plane to $z = f(x,y)$ at $P(2,4)$

By formula, $z = f(x_0, y_0) + \nabla f(x_0,y_0) \cdot (x-x_0,\ y-y_0)$

$\nabla f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$

$\frac{\partial f}{\partial x} = 9 - 3x^2$, $\frac{\partial f}{\partial y} = 2y - 7$

$\nabla f(2,4) = (-3, 1)$

$\therefore\ z = f(2,4) + (-3,1)\cdot(x-2,\ y-4)$

$= -2 - 3x + 6 + y - 4$

$= -3x + y - 6$ ※

b) to find stationary pts, let $\nabla f(x,y) = (0,0)$

from above part a)

$\frac{\partial f}{\partial x} = 9 - 3x^2 = 0 \Rightarrow x = \pm\sqrt{3}$

$\frac{\partial f}{\partial y} = 2y - 7 = 0 \Rightarrow y = \frac{7}{2}$

$f_{xx} = -6x$

$f_{xy} = 0$

$f_{yy} = 2$

i) pt $(\sqrt{3}, \frac{7}{2})$, $D = f_{xx}f_{yy} - f_{xy}^2 = -6\sqrt{3}(2) < 0$, saddle point

ii) pt $(-\sqrt{3}, \frac{7}{2})$, $D = 6\sqrt{3}(2) > 0$, $f_{xx} = 6\sqrt{3} > 0$, minimum point ※

c) $\frac{dx}{dt} = 6\text{cm/s}$, $\frac{dy}{dt} = -4\text{cm/s}$, $\frac{d\theta}{dt} = -0.05\text{ rad/s}$

Given $x = 80$, $y = 100$, $\theta = \pi/6$

Area of $\Delta = \frac{1}{2}xy\sin\theta$

$\frac{dA}{dt} = \frac{1}{2}\left[\frac{dx}{dt}y\sin\theta + x\frac{dy}{dt}\sin\theta + xy\cos\theta\frac{d\theta}{dt}\right]$ (product rule)

$\left.\frac{dA}{dt}\right|_{(80,100,\pi/6)} = \frac{1}{2}\left[6(100)\sin\frac{\pi}{6} + 80(-4)\sin\left(\frac{\pi}{6}\right) + 80(100)\cos\frac{\pi}{6}(-0.05)\right]$

$= 52.68\ \text{cm}^2/\text{s}$ ※

P②

4. a) $T(x,y,z) = 6x - 3y + 2z$, surface : $(x-1)^2 + 3y^2 + z^2 = 43$

By Lagrange Multiplier Method

$$\nabla T(x,y,z) = \lambda \nabla S(x,y,z)$$

Compare $\hat{i}$: $6 = 2\lambda(x-1)$ $\Rightarrow x-1 = \frac{3}{\lambda}$

$\hat{j}$: $-3 = 6\lambda y$ $\Rightarrow y = -\frac{1}{2\lambda}$

$\hat{k}$: $2 = 2\lambda z$ $\Rightarrow z = \frac{1}{\lambda}$ } solve

Given : $(x-1)^2 + 3y^2 + z^2 = 43$

$$\left(\frac{3}{\lambda}\right)^2 + 3\left(-\frac{1}{2\lambda}\right)^2 + \left(\frac{1}{\lambda}\right)^2 = 43$$

$$\lambda = \frac{1}{2}$$

$\therefore$ $x = \frac{3}{0.5} + 1 = 7$

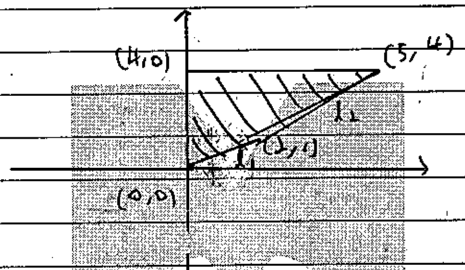
$y = -\frac{1}{2(0.5)} = -1$

$z = \frac{1}{0.5} = 2$

$T(7,-1,2) = 49\ °C$ ✱

b)

b)



$z(x,y) = x^2$

l_1 : $y = \frac{1}{2}x$ ✱

l_2 : $\frac{y-1}{x-2} = \frac{4-1}{5-2}$

$y = x - 2 + 1$

$= x - 1$ ✱

$V = \iint_R z(x,y)\, dA$

$= \iint_D x^2\, dy\, dx$

$= \int_0^2 \int_{x-1}^{4} x^2\, dy\, dx + \int_2^5 \int_{\frac{1}{2}x}^{4} x^2\, dy\, dx$

$= \int_0^2 \left[x^2 y\right]_{x-1}^{4} dx + \int_2^5 \left[x^2 y\right]_{\frac{1}{2}x}^{4} dx$

$= \int_0^2 4x^2 - x^3 + x^2\, dx + \int_2^5 4x^2 - \frac{1}{2}x^3\, dx$

$= \frac{28}{3} + \frac{639}{8}$

$= 89.208$ ✱

5 a) $0<a<1$, $\lim_{n\to\infty} \sqrt[n]{1+a^n}$

$1 < 1+a^n < 1+1^n$

$\sqrt[n]{1} < \sqrt[n]{1+a^n} < \sqrt[n]{2}$

$\lim_{n\to\infty} 1 < \lim_{n\to\infty} \sqrt[n]{1+a^n} < \lim_{n\to\infty} \sqrt[n]{2}$

$2^{\frac{1}{n}} = e^{\frac{1}{n}\ln 2}$

$\lim_{n\to\infty} \sqrt[n]{2} = \lim_{n\to\infty} e^{\left(\frac{1}{n}\ln 2\right)}$

$= e^{\lim_{n\to\infty} \frac{1}{n}\ln 2}$

$= e^{0}$

$= 1$

$1 < \lim_{n\to\infty} \sqrt[n]{1+a^n} < 1$

$\therefore$ By squeeze theorem, $\lim_{n\to\infty} \sqrt[n]{1+a^n} = 1$, whr $0 < a < 1$ #

b) $\lim_{n\to\infty} \sqrt[n]{3^n+7^n}$

$= \lim_{n\to\infty} \sqrt[n]{1+\left(\frac{3}{7}\right)^n} \times \sqrt[n]{7^n}$

$= 7(1)$

$= 7$ #

c) $\sum_{n=0}^{\infty} (3^n+7^n)x^n$

By root test

$\lim_{n\to\infty} \sqrt[n]{(3^n+7^n)(x^n)}$

$= \lim_{n\to\infty} \sqrt[n]{3^n+7^n} \lim_{n\to\infty} x$

$= 7x$

for convergent, $7x<1$, thus $x < \frac{1}{7}$ #

Sum $= \sum_{n=0}^{\infty} (3^n+7^n)x^n$

$= \sum_{n=0}^{\infty} (3^n+7^n) \sum_{n=0}^{\infty} x^n$, $|x| < \frac{1}{7} < 1$

$= \sum_{n=0}^{\infty} (3^n+7^n) \times \frac{1}{1-x}$

6 a) $\sum_{n=0}^{\infty} \frac{(-1)^n (n^2)}{5+n^3}$

Check $\sum_{n=0}^{\infty} \frac{(-1)^n (n^2)}{5+n^3}$

By Alternating Series Test

let $a_n = \frac{n^2}{5+n^3}$

① $\lim_{n\to\infty} \left(\frac{n^2}{5+n^3}\right) = \lim_{n\to\infty} \left(\frac{\frac{1}{n^2}}{\frac{5}{n^3}+1}\right) = 0$

② $\frac{(n+1)^2}{5+(n+1)^3} < \frac{n^2}{5+n^3}$

∴ from result ① & ②

∴ converges

Check $\sum_{n=0}^{\infty} \left| \frac{(-1)^n n^2}{5+n^3} \right|$

$= \sum_{n=0}^{\infty} \frac{n^2}{5+n^3}$

let $a_n = \frac{n^2}{5+n^3}$, $b_n = \frac{1}{n}$

By limit comparison test

$\lim_{n\to\infty} \left(\frac{\frac{n^2}{5+n^3}}{\frac{1}{n}} \right)$

$= \lim_{n\to\infty} \left(\frac{\frac{\frac{1}{n}}{1+\frac{5}{n^3}}}{\frac{1}{n}} \right)$

$= \lim_{n\to\infty} \frac{1}{1+\frac{5}{n^3}} = 1$

∴ Since $\sum_{n=0}^{\infty} b_n$ diverges, then $\sum_{n=0}^{\infty} a_n$ diverges

∴ Conditionally converges ##

b) $f(x) = \frac{1}{4+x-x^2}$, $a=0$

$f'(x) = -(4+x-x^2)^{-2}(1-2x)$

$f''(x) = -(4+x-x^2)^{-2}(-2) + (1-2x)[2(4+x-x^2)^{-3}(1-2x)]$

$f^3(x) = 2(-2)(4+x-x^2)^{-3}(1-2x) + (1+2x)^2(-6)(4+x-x^2)^{-4}(1+2x) + 2(4+x-x^2)^{-3}(2)(1-2x)(-2$

$f(0) = \frac{1}{4}$

$f'(0) = -\frac{1}{16}$

$f''(0) = \frac{5}{32}$

$f^3(0) = -\frac{27}{128}$

$T_3 = \frac{1}{4} - \frac{1}{16}x + \frac{5}{32}\frac{1}{2!}x^2 - \frac{27}{128}\frac{1}{3!}x^3$

let $4+x-x^2 = 4.09$

Solve, $x = \frac{1}{10} / \frac{9}{10}$

$\frac{1}{4.09} = T_3\big|_{x=\frac{9}{10}} = \frac{1}{4} - \frac{1}{16}\left(\frac{9}{10}\right) + \frac{5}{2\times 32}\left(\frac{9}{10}\right)^2 - \frac{27}{128}\left(\frac{9}{10}\right)^3$

$= 0.2314$ ##

$T_3\big|_{x=\frac{1}{10}} = \frac{1}{4} - \frac{1}{16}\left(\frac{1}{10}\right) + \frac{5}{64}\left(\frac{1}{10}\right)^2 - \frac{27}{128}\left(\frac{1}{10}\right)^3$

$= 0.2443$ ##

NANYANG TECHNOLOGICAL UNIVERSITY
SEMESTER 2 EXAMINATION 2016–2017
MH1811 – MATHEMATICS 2

APRIL 2017 TIME ALLOWED: 2 HOURS

Matriculation Number:

Seat Number:

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains SIX (6) questions and comprises THIRTEEN (13) pages.
2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
3. This is a **RESTRICTED OPEN BOOK** exam. Each candidate is allowed to bring a hand-written A4 help sheet.
4. Candidates may use calculators. However, they should write down systematically the steps in the workings.
5. All your solutions should be written in this booklet within the space provided after each question. However, if you write your solutions on other pages, please indicate them clearly.

For examiners only

Question	Marks
1 (15)	
2 (15)	
3 (15)	

Question	Marks
4 (15)	
5 (20)	
6 (20)	

TOTAL (100)

QUESTION 1. (15 Marks)

(a) Consider the function $f(x) = \sqrt{2-x^2}$.

(i) Complete the table for the derivatives and the Taylor coefficients, c_k, of the function $f(x)$ at $a = -1$, for $k = 0, 1, 2$.

k	$f^{(k)}(x)$	$f^{(k)}(-1)$	c_k
0	$f(x) = \sqrt{2-x^2}$		
1			
2			

(ii) Write down the Taylor polynomial $T_2(x)$ for $f(x)$ at $a = -1$.

(iii) Use the Taylor polynomial in Part (ii) to approximate the value $\sqrt{1.19}$.

Question 1 continues on page 3.

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(b) Is the series

$$\sum_{n=1}^{\infty} \cos(n\pi)\sin\left(\frac{\pi}{\sqrt{n}}\right)$$

absolutely convergent, conditionally convergent or divergent? Justify your answer.

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QUESTION 2.

(15 Marks)

Consider the power series $\sum_{n=1}^{\infty} \frac{n}{2^{n+3}}(x+1)^n$.

(a) Determine the **interval of convergence** of the power series $\sum_{n=1}^{\infty} \frac{n}{2^{n+3}}(x+1)^n$.

On the real number line in Figure 1, indicate the interval of convergence you have obtained.

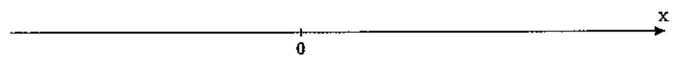

Figure 1.

Question 2 continues on page 5.

(b) Find the sum $s(x)$ of the power series $\sum_{n=1}^{\infty} \frac{n}{2^{n+3}}(x+1)^n$ for each x in the interval of convergence. Justify your answer.

(That is, find the function $s(x)$ such that $s(x) = \sum_{n=1}^{\infty} \frac{n}{2^{n+3}}(x+1)^n$.)

End of Question 2.

QUESTION 3. (15 Marks)

Consider the function f where

$$f(x,y) = 4 - x^2 - y.$$

(a) In Figure 2, sketch the following

(i) the level curve $f(x,y) = 0$.

(ii) the level curve $f(x,y) = k$ which passes through the point $P(-1,1)$. What is the value of the constant k?

Value of k : ______________________

(iii) the direction **u** at $P(-1,1)$ in which f increases most rapidly. What is the unit vector **u**?

Vector **u** : ______________________

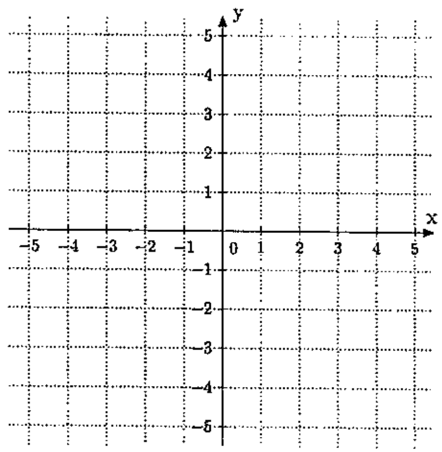

Figure 2.

Question 3 continues on page 7.

(b) Use total differential to estimate the change in f from $P(-1,1)$ to $Q(-1.02, 0.97)$.

(c) Find the **maximum value** of $H(x,y) = 100 - 6x + xy$ on the closed and bounded region enclosed by the level curve $f(x,y) = 0$ (in Part (a)) and $y = 0$. Justify your answer.

End of Question 3.

QUESTION 4. (15 Marks)

(a) Consider a lamina which can be described by the triangular region with vertices $(0,0)$ and $(4,1)$ and $(9,1)$ on an xy-plane. The density (in grammes per unit square) of the lamina is

$$\sigma(x,y) = xy^2 + \sqrt{x}.$$

Evaluate the mass (in grammes) of the lamina. Leave your answer up to 2 decimal places.

Question 4 continues on page 9.

(b) Determine all stationary points of the function $f(x, y) = (y^2 - 3y)\sin x$, in the open rectangle $0 < x < 2\pi$ and $1 < y < 5$, and classify each of them as a local maximum, local minimum or saddle point.

End of Question 4.

QUESTION 5. (20 Marks)

(a) Solve the following initial value problem.

$$y' + 2xy = e^{-x^2} + x, \quad y(0) = 2.$$

Express y in terms of x explicitly.

Question 5 continues on page 11.

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(b) By letting $y = Vx$ where V is a function of x, solve the following differential equation

$$y' = \frac{9x^3 + 2x^2y + 4y^3}{x^3 + 3xy^2}, \quad x > 0, y > 0.$$

(You may leave the general solution in implicit form.)

End of Question 5.

QUESTION 6. **(20 Marks)**

(a) Verify that the following differential equation is **exact** and proceed to solve it.

$$\left(\frac{1}{\sqrt{x-1}} + e^x \sin(\pi y)\right) dx + \left(\sec^2(\pi y) + \pi e^x \cos(\pi y)\right) dy = 0, \quad x > 1,\ |y| < 0.5.$$

Question 6 continues on page 13.

(b) Solve the second order differential equation

$$y'' - 5y' = 45x^2 + 2x.$$

END OF PAPER

MH1811 April 2017

① (a) i) $k = 0$

$$f^{(0)}(x) = \sqrt{2-x^2} \rightarrow f^{(0)}(-1) = \sqrt{2-(-1)^2} = 1$$

$$C_0 = \frac{f^{(0)}(-1)}{0!} = \frac{1}{1} = 1$$

$k = 1$

$$f^{(1)}(x) = \frac{-2x}{2\sqrt{2-x^2}} = \frac{-x}{\sqrt{2-x^2}} \rightarrow f^{(1)}(-1) = \frac{1}{\sqrt{2-1}} = 1$$

$$C_1 = \frac{f^{(1)}(-1)}{1!} = \frac{1}{1} = 1$$

$k = 2$

$$f^{(2)}(x) = \frac{-\sqrt{2-x^2} - \left(\frac{-2x}{2\sqrt{2-x^2}}\right)(-x)}{2-x^2} = \frac{-\sqrt{2-x^2} - \frac{x^2}{\sqrt{2-x^2}}}{2-x^2}$$

$$f^{(2)}(-1) = \frac{-\sqrt{2-1} - \frac{1}{\sqrt{2-1}}}{2-1} = -2$$

$$C_2 = \frac{f^{(2)}(-1)}{2!} = \frac{-2}{2} = -1$$

ii)
$$T_2(x) = 1 + 1(x-(-1)) - 1(x-(-1))^2$$
$$= 1 + x + 1 - (x^2+2x+1)$$
$$= -x^2 - x + 1$$

iii)
$$f(x) = \sqrt{2-x^2} = \sqrt{1.19}$$
$$\sqrt{2-x^2} = \sqrt{2-0.81}$$
$$x^2 = 0.81 \rightarrow x = 0.9 \vee -0.9$$

We choose $x = -0.9$ as it is closer to $a = -1$

$$f(-0.9) \approx T_2(-0.9) = -(-0.9)^2 - (-0.9) + 1$$
$$= 1.09$$

(b) For $n \geq 1$; $\cos(n\pi) = (-1)^n$

$$\sum_{n=1}^{\infty} \cos(n\pi)\sin\left(\frac{\pi}{\sqrt{n}}\right) = \sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{\pi}{\sqrt{n}}\right)$$

Let $a_n = (-1)^n \sin\frac{\pi}{\sqrt{n}}$. Then $|a_n| = \sin\frac{\pi}{\sqrt{n}}$

Apply limit comparison test with $\sum_{n=1}^{\infty} \frac{\pi}{\sqrt{n}}$ which is a divergent series

$$\lim_{n\to\infty} \frac{|a_n|}{\frac{\pi}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\sin\left(\frac{\pi}{\sqrt{n}}\right)}{\frac{\pi}{\sqrt{n}}}$$

$$= \lim_{\frac{\pi}{\sqrt{n}}\to 0} \frac{\sin\left(\frac{\pi}{\sqrt{n}}\right)}{\frac{\pi}{\sqrt{n}}} = 1$$

By limit comparison test, both $\sum_{n=1}^{\infty} \frac{\pi}{\sqrt{n}}$ and $\sum_{n=1}^{\infty} |a_n|$ are divergent.

Proceed to determine the convergence of $\sum_{n=1}^{\infty} (-1)^n \sin\frac{\pi}{\sqrt{n}}$ by alternating series test

- For $n \geq 1$, the term $\frac{\pi}{\sqrt{n}}$ is decreasing with $0 < \frac{\pi}{\sqrt{n}} \leq \pi$
  Hence the term $\frac{\pi}{\sqrt{n}}$ is always in first or second quadrant and $\sin\frac{\pi}{\sqrt{n}}$ is always positive.
- For $n \geq 4$, $\sin\frac{\pi}{\sqrt{n}}$ is always decreasing $\left(\frac{\pi}{2} \geq \frac{\pi}{\sqrt{n}} > 0\right)$
- $\lim_{n\to\infty} \sin\frac{\pi}{\sqrt{n}} = \sin(0) = 0$

Hence by AST, the series $\sum_{n=4}^{\infty} (-1)^n \sin\frac{\pi}{\sqrt{n}}$ is convergent and so is the series $\sum_{n=1}^{\infty} (-1)^n \sin\frac{\pi}{\sqrt{n}}$

$\therefore$ The series $\sum_{n=1}^{\infty} \cos(n\pi)\sin\left(\frac{\pi}{\sqrt{n}}\right)$ is conditionally convergent

(2) (a) Let $a_n = \frac{n}{2^{n+3}}(x+1)^n$

$$= \frac{n}{8}\left(\frac{x+1}{2}\right)^n$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\frac{n+1}{8}\left|\frac{x+1}{2}\right|^{n+1}}{\frac{n}{8}\left|\frac{x+1}{2}\right|^{n}} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)\left|\frac{x+1}{2}\right|$$

$$= \left|\frac{x+1}{2}\right| \lim_{n\to\infty}\left(\frac{n+1}{n}\right)$$

$$= \left|\frac{x+1}{2}\right|$$

$\therefore$ By ratio test, $\sum_{n=1}^{\infty} \frac{n}{2^{n+3}}(x+1)^n$ converges when

$$\left|\frac{x+1}{2}\right| < 1 \iff |x+1| < 2$$

$$\iff -2 < x+1 < 2$$

$$\iff -3 < x < 1$$

and diverges otherwise

We proceed to check the divergence when $x = -3$ and $x = 1$

When $x = -3$

The series becomes $\sum_{n=1}^{\infty} \frac{n}{8}(-1)^n$ which is an alternating series

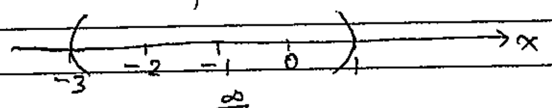
Let $b_n = \frac{n}{8}$; $\lim_{n\to\infty} \frac{n}{8} = \infty$

$\therefore$ Hence, the series diverges when $x = -3$

When $x = 1$

The series becomes $\sum_{n=1}^{\infty} \frac{n}{8}$ which is a divergent series

$\therefore$ Interval of convergence : $-3 < x < 1$



(b) $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$ ← known

$$\frac{1}{1-\left(\frac{x+1}{2}\right)} = \sum_{n=0}^{\infty} \left(\frac{x+1}{2}\right)^n \quad ; \text{ for } \left|\frac{x+1}{2}\right| < 1$$

$$\frac{2}{1-x} = \sum_{n=0}^{\infty} \left(\frac{x+1}{2}\right)^n$$

differentiating → $\frac{2}{(1-x)^2} = \sum_{n=1}^{\infty} n\left(\frac{x+1}{2}\right)^{n-1}\left(\frac{1}{2}\right)$

$$\frac{4}{(1-x)^2} = \sum_{m=0}^{\infty} (m+1)\left(\frac{x+1}{2}\right)^m$$

← m and n are dummy variables

$$\frac{4}{(1-x)^2} = \sum_{n=0}^{\infty} n\left(\frac{x+1}{2}\right)^n + \sum_{n=0}^{\infty} \left(\frac{x+1}{2}\right)^n$$

$$\frac{4}{(1-x)^2} = \sum_{n=0}^{\infty} n\left(\frac{x+1}{2}\right)^n + \frac{2}{1-x}$$

$$\frac{4}{(1-x)^2} - \frac{2}{1-x} = \sum_{n=0}^{0} n\left(\frac{x+1}{2}\right)^n + \sum_{n=1}^{\infty} n\left(\frac{x+1}{2}\right)^n$$

$$\frac{4}{(1-x)^2} - \frac{2}{1-x} = 0 + 8\sum_{n=1}^{\infty} \frac{n}{8}\left(\frac{x+1}{2}\right)^n$$

$$\frac{1}{2(1-x)^2} - \frac{1}{4(1-x)} = \sum_{n=1}^{\infty} \frac{n}{8}\left(\frac{x+1}{2}\right)^n$$

$$\therefore \sum_{n=1}^{\infty} \frac{n}{2^{n+3}}(x+1)^n = \frac{1}{2(1-x)^2} - \frac{1}{4(1-x)}$$

(3) (a) i) $f(x,y) = 4 - x^2 - y = 0$

$\iff y = 4 - x^2$

Intercepts x axis at $(-2,0)$ and $(2,0)$

Intercepts y axis at $(0,4)$

ii) $k = 4 - (-1)^2 - 1 = 2$

$f(x,y) = 4 - x^2 - y = 2$

$\iff y = 2 - x^2$

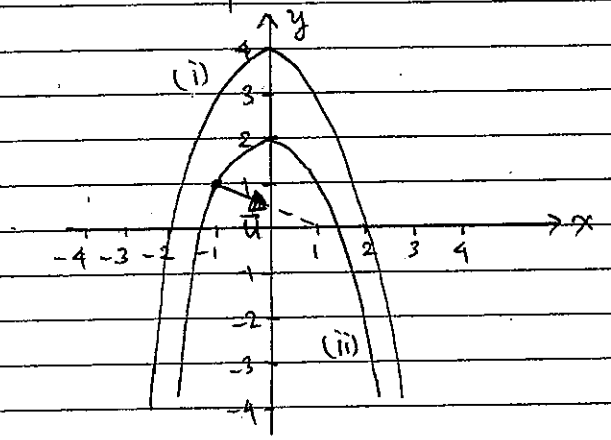
Intercepts x axis at $(-\sqrt{2},0)$ and $(\sqrt{2},0)$

Intercepts y axis at $(0,2)$

iii) $\nabla f(x,y) = (-2x, -1)$

$\nabla f(-1,1) = (2,-1)$

$\vec{u} = \dfrac{\nabla f(-1,1)}{\|\nabla f(-1,1)\|} = \dfrac{(2,-1)}{\sqrt{2^2+1^2}} = \dfrac{1}{\sqrt{5}}(2,-1)$



(b) $\Delta f \approx \nabla f(-1,1) \cdot (-1.02 - (-1), 0.97 - 1)$

$= (2,-1) \cdot (-0.02, -0.03)$

$= -0.01$

(c) i) Along $f(x,y) = 0 \iff y = 4 - x^2$

$H(x,y) = 100 - 6x + x(4 - x^2)$

$= -x^3 - 2x + 100 = g(x)$

$\dfrac{dH}{dx} = -3x^2 - 2 = 0 \iff x^2 = -\dfrac{2}{3}$ (no real solution)

Check endpoints

$H(-2,0) = 100 - 6(-2) = 112$ → maximum

$H(2,0) = 100 - 6(2) = 88$

ii) Along $y = 0$

$H(x,y) = 100 - 6x \rightarrow$ linear function

Proceed to check endpoints only

$H(-2,0) = 112 \rightarrow$ maximum

$H(2,0) = 88$

iii) Inside the region $\rightarrow$ use Lagrange multiplier

$\nabla H(x,y) = (-6+y, x)$

$\nabla H(x,y) = \lambda \nabla f(x,y)$

(1) : $y - 6 = \lambda(-2x)$

(2) : $x = \lambda(-1) \iff x = -\lambda$

(3) : $y = 4 - x^2$

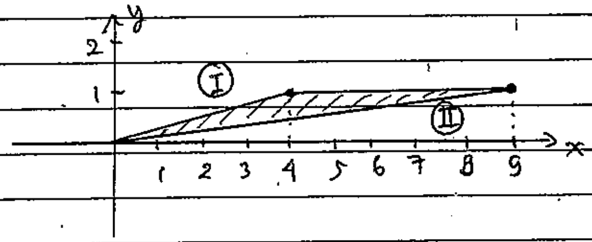
(1) $\rightarrow$ $y - 6 = \lambda(2\lambda) \iff y = 2\lambda^2 + 6$

(3) $\rightarrow$ $2\lambda^2 + 6 = 4 - \lambda^2$

$3\lambda^2 = -2 \rightarrow$ no real solution

$\therefore$ Maximum value of $H(x,y) = 112$ at $(-2,0)$

(4) (a)



I: $\frac{y-0}{1-0} = \frac{x-0}{4-0}$

$y = \frac{1}{4}x \iff x = 4y$

II: $\frac{y-0}{1-0} = \frac{x-0}{9-0}$

$y = \frac{1}{9}x \iff x = 9y$

$4y \le x \le 9y$

$0 \le y \le 1$

$m = \iint \sigma(x,y)\, dx\, dy$

$= \int_0^1 \int_{4y}^{9y} (xy^2 + \sqrt{x})\, dx\, dy = \int_0^1 \left[\frac{1}{2}x^2y^2 + \frac{2}{3}x^{\frac{3}{2}}\right]_{4y}^{9y}$

$= \int_0^1 \left(\frac{81}{2}y^4 + 18y^{\frac{3}{2}} - 8y^4 - \frac{16}{3}y^{\frac{3}{2}}\right) dy = \int_0^1 \left(\frac{65}{2}y^4 + \frac{38}{3}y^{\frac{3}{2}}\right) dy$

$= \left[\frac{13}{2}y^5 + \frac{76}{15}y^{\frac{5}{2}}\right]_0^1$

$= \frac{13}{2} + \frac{76}{15} - 0 - 0 = 11.56$ grams

(b) $f_x(x,y) = (y^2-3y)\cos x$; $f_y(x,y) = (2y-3)\sin x$

$f_{xx}(x,y) = -\sin x\,(y^2-3y)$; $f_{yy}(x,y) = 2\sin x$

$f_{xy}(x,y) = (2y-3)\cos x = f_{yx}(x,y)$

Stationary points : $\nabla f(x,y) = (0,0)$

(1) : $(y^2-3y)\cos x = 0$, and

(2) : $(2y-3)\sin x = 0$

(1) : $(y^2-3y)\cos x = 0$

$y^2-3y = 0 \quad \vee \quad \cos x = 0$

$y=0 \vee y=3 \quad \vee \quad x = (n-\frac{1}{2})\pi$; $n = 1,2,3,\dots$

Since $0 < x < 2\pi$ and $1 < y < 5$

From first equation we get $y = 3 \vee x = \frac{\pi}{2} \vee x = \frac{3\pi}{2}$

(2) : $(2y-3)\sin x = 0$

$2y-3 = 0 \quad \vee \quad \sin x = 0$

$y = \frac{3}{2} \quad\quad x = (n-1)\pi$; $n = 1,2,3,\dots$

From 2nd equation we get $y = \frac{3}{2} \vee x = \pi$

$D(x,y) = f_{xx} f_{yy} - (f_{xy})^2$

$= -2(y^2-3y)\sin^2 x - (2y-3)^2\cos^2 x$

Stationary points : $\{(\frac{\pi}{2},\frac{3}{2}), (\frac{3\pi}{2},\frac{3}{2}), (\pi,3)\}$

(1) : $(\frac{\pi}{2},\frac{3}{2}) \to D(\frac{\pi}{2},\frac{3}{2}) = -2(\frac{9}{4}-\frac{9}{2})\sin^2\frac{\pi}{2} - (3-3)^2\cos^2\frac{\pi}{2}$

$= \frac{9}{2}$

$f_{xx}(\frac{\pi}{2},\frac{3}{2}) = -\sin\frac{\pi}{2}(\frac{9}{4}-\frac{9}{2}) = \frac{9}{4}$

↳ local min

(2) : $(\frac{3\pi}{2},\frac{3}{2}) \to D(\frac{3\pi}{2},\frac{3}{2}) = -2(\frac{9}{4}-\frac{9}{2})\sin^2\frac{3\pi}{2} - (3-3)^2\cos^2\frac{3\pi}{2}$

$= \frac{9}{2}$

$f_{xx}(\frac{3\pi}{2},\frac{3}{2}) = -\sin\frac{3\pi}{2}(\frac{9}{4}-\frac{9}{2}) = -\frac{9}{4}$

↳ local max

(3) : $(\pi,3) \to D(\pi,3) = -2(9-9)\sin^2\pi - (6-3)^2\cos^2\pi$

$= -9$

↳ saddle point

(5) (a) $y' + 2xy = e^{-x^2} + x$

$I = e^{\int 2x\,dx} = e^{x^2}$

$y = \frac{1}{I}\int I\,(e^{-x^2} + x)\,dx$

$= e^{-x^2}\int e^{x^2}(e^{-x^2} + x)\,dx$

$= e^{-x^2}\int (1 + xe^{x^2})\,dx$

$y = e^{-x^2}(x + \frac{1}{2}e^{x^2} + c)$

$y(0) = 2 = e^0(0 + \frac{1}{2}e^0 + c)$

$2 = \frac{1}{2} + c \iff c = \frac{3}{2}$

$y = xe^{-x^2} + \frac{1}{2} + \frac{3}{2}e^{-x^2}$

(b) $y' = \dfrac{9 + 2(\frac{y}{x}) + 4(\frac{y}{x})^3}{1 + 3(\frac{y}{x})^2}$

$V'x + V = \dfrac{9 + 2V + 4V^3}{1 + 3V^2}$

$V'x = \dfrac{9 + 2V + 4V^3 - V - 3V^3}{1 + 3V^2}$

$\dfrac{dV}{dx}x = \dfrac{V^3 + V + 9}{1 + 3V^2}$

$\int \dfrac{1 + 3V^2}{V^3 + V + 9}\,dV = \int \dfrac{dx}{x}$

$\ln|V^3 + V + 9| = \ln|x| + C_0$; C_0 is an arbitrary constant

$V^3 + V + 9 = Cx$; $C = \pm e^{C_0} \neq 0$

$(\frac{y}{x})^3 + \frac{y}{x} + 9 = Cx$

$\therefore\ y^3 + x^2y + 9x^3 - Cx^4 = 0$; C is an arbitrary constant $\neq 0$

(6) (a) $M = \frac{1}{\sqrt{x-1}} + e^x\sin(\pi y)$

$N = \sec^2(\pi y) + \pi e^x\cos(\pi y)$

$M_y = \pi e^x\cos(\pi y)$

$N_x = \pi e^x\cos(\pi y)$

$\} M_y = N_x \rightarrow$ exact D.E

$\Rightarrow \dfrac{\partial u}{\partial x} = \dfrac{1}{\sqrt{x-1}} + e^x\sin(\pi y)$

$u = 2\sqrt{x-1} + e^x\sin(\pi y) + g(y)$

~~$\frac{\partial u}{\partial y} = \pi e^x \cos \pi y + g'(y) = \pi e^x \cos \pi y + \sec^2 \pi y$~~

Hence $g'(y) = \sec^2 \pi y$

$g(y) = \frac{1}{\pi} \tan(\pi y)$

Potential function $u(x,y) = 2\sqrt{x-1} + e^x \sin \pi y + \frac{1}{\pi} \tan \pi y$

General solution : $2\sqrt{x-1} + e^x \sin \pi y + \frac{1}{\pi} \tan \pi y = C$

where C is an arbitrary constant

(b) Homogeneous equation : $y'' - 5y' = 0$

Characteristic equation : $\lambda^2 - 5\lambda = 0$ $\quad \lambda_1 = 0,\ \lambda_2 = 5$

$y_h = C_1 e^0 + C_2 e^{5x} = C_1 + C_2 e^{5x}$

where C_1 and C_2 are arbitrary constants

$y_p = A + Bx + Cx^2$

$y_p' = B + 2Cx$

$y_p'' = 2C$

Substituting into the D.E

$(2C) - 5(B + 2Cx) = 45x^2 + 2x$

(Left side is lacking the term x^2, find another y_p)

$y_p = A + Bx + Cx^2 + Dx^3$

$y_p' = B + 2Cx + 3Dx^2$

$y_p'' = 2C + 6Dx$

Substituting into D.E

$(2C + 6Dx) - 5(B + 2Cx + 3Dx^2) = 45x^2 + 2x$

$-15Dx^2 + (6D - 10C)x + (2C - 5B) = 45x^2 + 2x$

$x^2 : \ -15D = 45 \iff D = -3$

$x : \ 6(-3) - 10C = 2 \iff C = -2$

$x^0 : \ 2(-2) - 5B = 0 \iff B = -\frac{4}{5}$

$y_p = -\frac{4}{5}x - 2x^2 - 3x^3$

General solution

$y = C_1 + C_2 e^{5x} - \frac{4}{5}x - 2x^2 - 3x^3$

(The term A in y_p can be included to C_1)